Incircular nets and confocal conics*

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1 Introduction

In this article we construct some grids naturally related with conics and quadrics in $\mathbb{R}^3$, and in particular with confocal conics.

We consider nets build by congruences of straight lines in the plane with the combinatorics of the square grid such that all corresponding coordinate quadrilaterals possess inscribed circles (incircular or IC-nets). It follows from the Graves-Chasles theorem that the vertices of an IC-net lie on confocal conics. We discuss a construction and geometric properties of these nets.

Our main new results are on checkerboard IC-nets in the plane. These are congruences of straight lines in the plane with the combinatorics of the square grid, combinatorially colored as a checkerboard, such that all black coordinate quadrilaterals possess inscribed circles. We show how this larger class of IC-nets appears quite naturally in Laguerre geometry of oriented planes and spheres and leads to new remarkable incidence theorems. Most of our results are valid in hyperbolic and spherical geometries as well. We present also generalizations in spaces of higher dimension, called checkerboard IS-nets (nets of hyperplanes such that all black coordinate hypercubes possess inscribed spheres). The construction of these nets is based on a new 9 inspheres incidence theorem.

Note that if we start from Poncelet polygons closed between two confocal ellipses, and continue all sides of this polygon, we get a Poncelet grid for confocal ellipses (see Schwartz [11] and M. Levi and S. Tabachnikov [10]). When the corresponding construction is cyclic, IC-net is a part of the Poncelet grid as in Fig. 1.

By $\mathbb{P}$ we denote a rectangle in $\mathbb{Z}^2$. Rectangles $m \times n$ are denoted by $\mathbb{P}_{m,n}$.

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2 Incircular (IC) nets

2.1 Main theorem

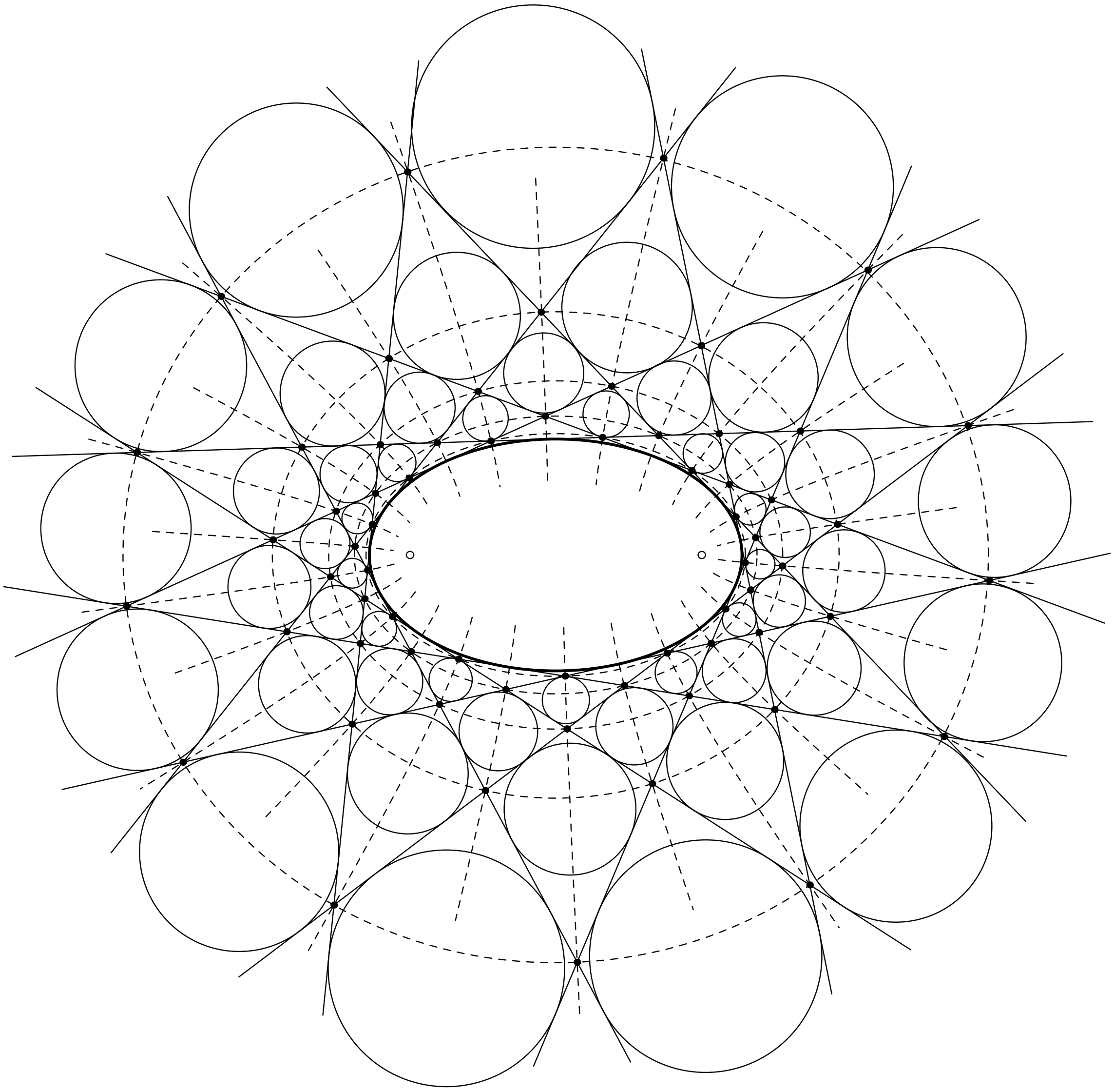


Figure 1: A Poncelet IC-net

We consider maps of the square grid to the plane $f : \mathbb{Z}^2 \to \mathbb{R}^2$ and use the following notations:

- $f_{i,j} = f(i,j)$ for the vertices of the net,
- $\Box^c_{i,j}$ for the quadrilateral $(f_{i,j}, f_{i+c,j}, f_{i+c,j+c}, f_{i,j+c})$ which we call a *net-square*,
- $\Box_{i,j}$ for the net-square $\Box^1_{i,j}$ which we call a *unit net-square*.

Definition 1. An IC-net (inscribed circular net) is a map $f : \mathbb{P} \to \mathbb{R}^2$ satisfying the following conditions:

(i) For any integer i the points $\{f(i,j)|j \in \mathbb{Z}\}$ lie on a straight line preserving the order, i.e the point $f(i,j)$ lies between $f(i,j-1)$ and $f(i,j+1)$. The same holds for points $\{f(i,j)|i \in \mathbb{Z}\}$. We call these lines the *lines of the IC-net.*

(ii) all unit net-squares $\square_{i,j}$ are circumscribed. We denote the inscribed circle of $\square_{i,j}$ by $\omega_{i,j}$ and its center by $o_{i,j}$.

An example of an IC-net is presented in Fig. 1. IC-nets have remarkable geometric properties which we summarize in the following theorem.

Theorem 2.1. *Let f be an IC-net. Then the following properties hold:*

(i) All lines of the IC-net f touch some conic α (possibly degenerate).

(ii) The points $f_{i,j}$, where $i+j = \text{const}$ lie on a conic confocal with α. As well the points $f_{i,j}$, where $i-j = \text{const}$ lie on a conic confocal with α.

(iii) All net-squares of f are circumscribed.

(iv) In any net-square with even sides the midlines have equal lengths:

$$|f_{i-c,j}f_{i+c,j}| = |f_{i,j-c}f_{i,j+c}|. \tag{1}$$

(v) The cross ratio

$$cr(f_{i,j_1}, f_{i,j_2}, f_{i,j_3}, f_{i,j_4}) = \frac{(f_{i,j_1} - f_{i,j_2})(f_{i,j_3} - f_{i,j_4})}{(f_{i,j_2} - f_{i,j_3})(f_{i,j_4} - f_{i,j_1})}$$

is independent of i. The cross ratio $cr(f_{i_1,j}, f_{i_2,j}, f_{i_3,j}, f_{i_4,j})$ is independent of j.

(vi) Consider the conics C_k that contain the points $f_{i,j}$ with $i+j=k$ (see (ii)). Then for any $l \in \mathbb{Z}$ there exists an affine transformation $A_{k,l} : C_k \to C_{k+2l}$ such that $A_{k,l}(f_{i,j}) = f_{i+l,j+l}$. The same holds for the conics through the points $f_{i,j}$ with $i-j = \text{const}$.

(vii) The net-squares $\square^c_{i,j}$ and $\square^{c+2l}_{i-l,j-l}$ are perspective.

(viii) Let $\omega_{i,j}$ be the inscribed circle of the unit net-square $\square_{i,j}$. Consider the cone in $\mathbb{R}^3$ intersecting the plane along $\omega_{i,j}$ at constant oriented angle (all the apexes $a_{i,j}$ of these cones lie in one half-space). Then all the apexes $a_{i,j}$ lie on one-sheeted hyperboloid.

(ix) Let $o_{i,j}$ be the center of the circle $\omega_{i,j}$. Then all $o_{i,j}$ with $i+j = \text{const}$ lie on a conic, and $o_{i,j}$ with $i-j = \text{const}$ also lie on a conic.

(x) The centers $o_{i,j}$ of circles of an IC-net build a projective image of an IC-net.

We will prove this theorem in Section 2.4. But before this we present some important facts about pencils of conics.

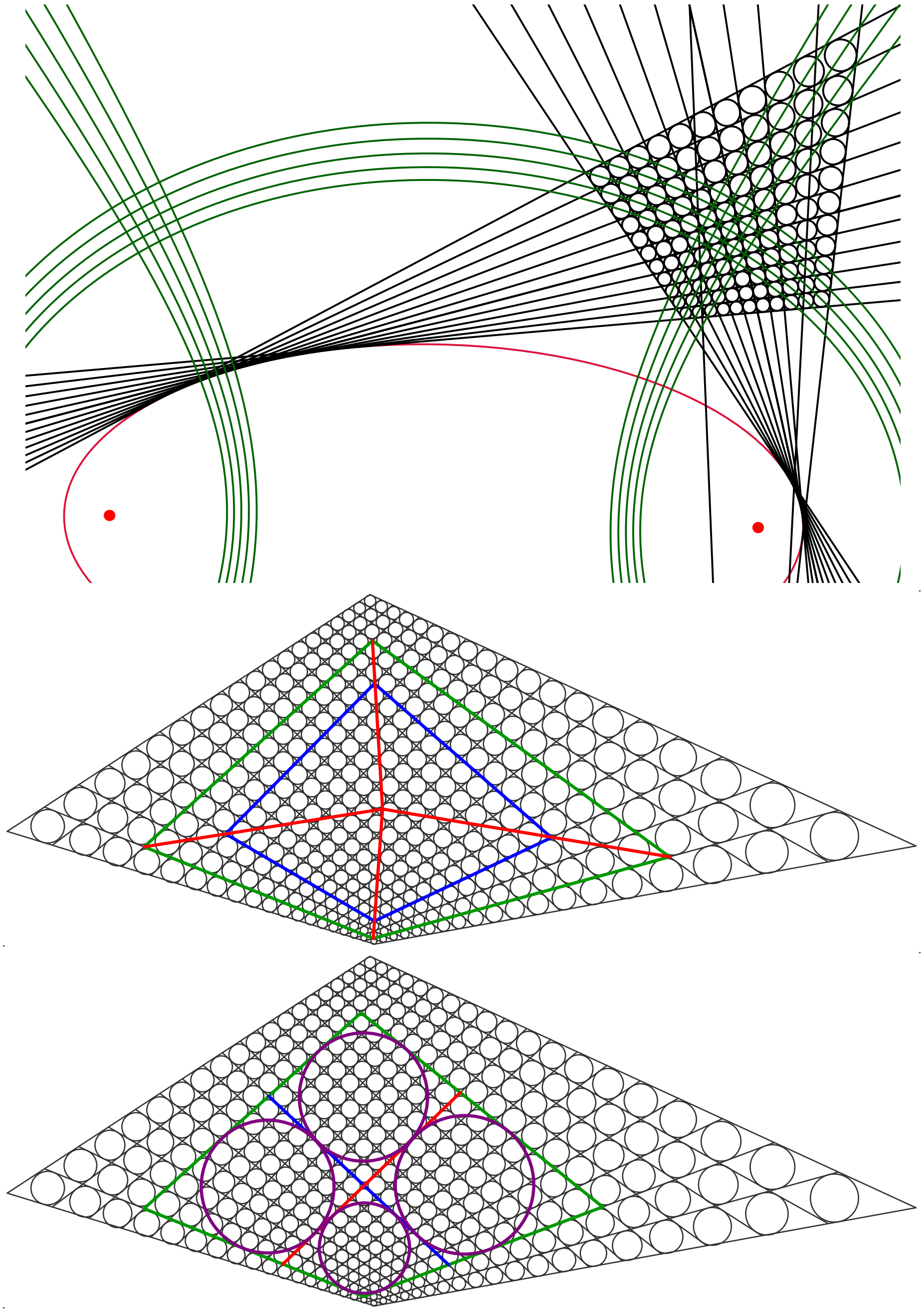


Figure 2: Geometry of IC-nets: (top) tangent lines and confocal conics, (middle) perspective net-squares, (bottom) circumscribed net-squares and equal length mid-lines

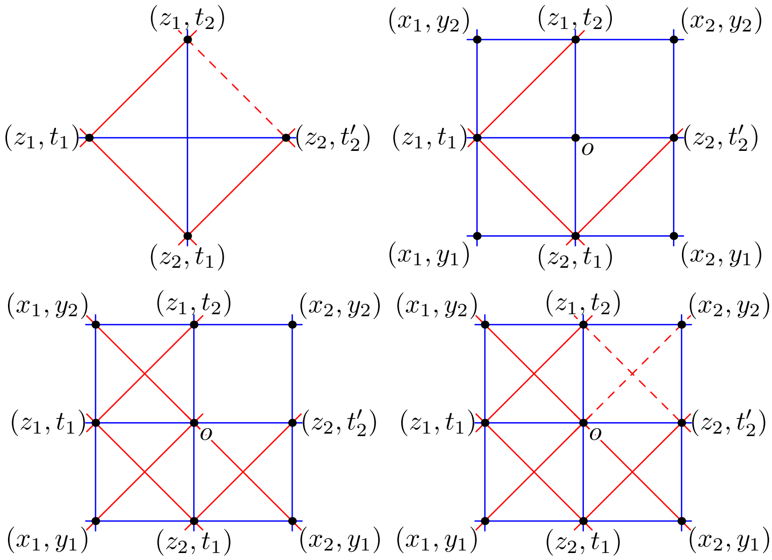


Figure 3: Two coordinate systems

2.2 The Graves–Chasles theorem

In this section we present a remarkable theorem of Graves–Chasles. It plays a crucial role in the proof of our main theorem and in construction of IC-nets. Although Darboux calls this theorem beautiful (see [7], p. 174, where he gives neither a proof nor the precise reference). It is of course not difficult to prove it by direct computation. However we chose another, more geometric approach. An advantage of our approach is that the proof of the corresponding theorem is valid for the hyperbolic space and for the sphere as well.

Lemma 2.2. *Suppose on a domain $\Omega \subset \mathbb{R}^2$ two coordinate systems (x, y) and (z, t) are given. Then the following two properties are equivalent:*

(i) if the points (x_1, y_1) and (x_2, y_2) have a same z coordinates, then the points (x_1, y_2) and (x_2, y_1) have the same t coordinate.

(ii) if the points (z_1, t_1) and (z_2, t_2) have a same x coordinates, then the points (z_1, t_2) and (z_2, t_1) have the same y coordinate.

Factorizing by reparametrizations $(x, y) \to (\phi(x), \psi(y))$ one obtains a coordinate line system.

Proof. Take two points (z_1, t_2) and (z_2, t_1) with the same x-coordinate and consider the line with the fixed y-coordinate passing through the point (z_1, t_1) (see Fig. 3). Let (z_2, t_2') be the point on this line with the coordinate $z = z_2$. To prove (i)⇒(ii) we have to show that $t_2' = t_2$.

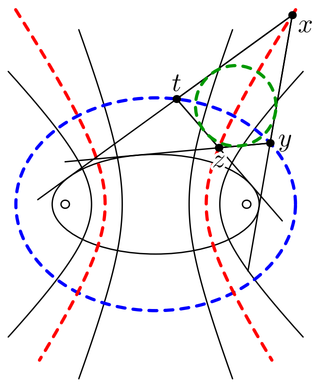


Figure 4: Graves–Chasles theorem

Draw the x, y coordinate lines through the points (z_1, t_1), (z_2, t_1), (z_1, t_2), and (z_2, t_2'), and denote the points of their intersection as shown in Fig. 3. Applying the property (i) for the pairs of points $\{(z_1, t_1), (z_2, t_1)\}$, $\{(z_1, t_1), (z_1, t_2)\}$, $\{(z_2, t_1), (z_2, t_2')\}$, we see that the points (x_1, y_2), o, and (x_2, y_1) have equal t-coordinates. From (i) we get that the points (x_1, y_1), o and (x_2, y_2) have equal z-coordinates. Finally, applying (i) to $\{o, (x_2, z_2)\}$, we obtain that $t_2 = t_2'$. □

Definition 2. We call a pair of coordinate line systems *diagonal-connected* if they satisfy the condition of Lemma 2.2.

We start with the following well-known lemma (for the proof see [3, Lemma 3.8] and [4, 16.6.4]).

Lemma 2.3. *Let a, b, c, d be four points on a conic α and the lines (ab) and (cd) touch some other conic β. Then lines (bd) and (ac) touch some conic γ from the pencil generated by the conics α and β. Moreover the tangent points of β with (ab) and (cd), and the tangent points of γ with (bd) and (ac) are collinear.*

Recall that a dual pencil of conics in $\mathbb{R}P^2$ is a set of conics with four common tangent lines (possibly complex). It is projectively dual to a pencil of conics (see [3] and [4]). The projectively dual form of Lemma 2.3 reads as follows.

Lemma 2.4. *Consider two coordinate systems in $\Omega \subset \mathbb{R}^2$ formed by a dual pencil of conics and the family of lines tangent to some conic from this pencil respectively. These coordinate line systems are diagonal-connected.*

Moreover, let the lines of the sides of the quadrilateral $(abcd)$ touch a conic α and its vertices a and c lie on conic β, the vertices b and d lie on a conic γ, and all three conics α, β, γ are from a dual pencil. Then the tangent lines through a and c to the conic β, and though b and d to the conic γ intersect in one point.

Theorem 2.5 (Graves–Chasles theorem)**.** *Suppose that the lines of all sides of the quadrilateral $(abcd)$ touch a conic α. Then the following three properties are equivalent:*

(i) $(abcd)$ *is circumscribed;*

(ii) Points a *and* c *lie on a conic confocal with* α*;*

(iii) Points b *and* d *lie on a conic confocal with* α*.*

Proof. Note that a family of confocal conics forms a dual pencil. Therefore the equivalence of conditions (ii) and (iii) follows directly from the first part of Lemma 2.4.

Let us show how the statement (i) follows from the other two. Assume that the quadrilateral $(abcd)$ satisfies conditions (ii) and (iii). Then the second part of Lemma 2.4 implies that the tangent lines to the conics at the vertices of $(abcd)$ intersect in a point. It remains to show that these tangent lines are bisectors of the quadrilateral $(abcd)$.

The classical equal angle lemma (see, for example, [3], ...) states that the bisectors of the angle formed by the tangent lines from a point p to the conic α coincide with the bisectors of the angles $\angle f_1 p f_2$, where f_1 and f_2 are the foci of the conic α. On the other hand the optical property of conics implies that this bisector is tangent to the conic confocal with α passing through p. This completes the implication (ii)$\Rightarrow$(i).

For the proof in the opposite direction (i)$\Rightarrow$(ii), we use the uniqueness of the configuration. Indeed, chose a point c' on the line (bc) such that a and c' lie on a conic from our confocal family, and point d' on the line (ad) such that $(c'd')$ is tangent to α. We proved already that then $(abc'd')$ is circumscribed. Moreover its incircle coincides with the incircle of the quadrilateral $(abcd)$ because it is uniquely determined by the lines (ad), (ab), and (bc). The incircle and the conic α have no more than four common tangent lines. Since they already have three common tangent lines (ad), (ab), and (bc), the common tangent lines (bc) and $(b'c')$ coincide. □

We will use also the following direct corollary of this Theorem.

Corollary 2.6. *Let the lines* $(ab), (bc), (cd)$ *of three sides of a circumscribed quadrilateral* $(abcd)$ *touch a conic* α *and the vertices* a, b *lie on a conic confocal to* α*. Then the line* (ad) *of the forth side also touches* α*.*

Remark 1. Dual pencils of conics are studied in particular in [2]. From Lemma 1 of this paper it follows that if the lines of the sides of a quadrilateral $(abcd)$ touch a conic α and a circle with center o, then the lines of the sides of the quadrilateral $f_1 a f_2 c$, where f_1 and f_2 are the foci of the conic α, are also tangent to a circle with center o. However circumscribity is equivalent to the fact that a and c lie on a conic with foci f_1 and f_2. That shows the equivalence (i)$\Leftrightarrow$(ii).

Applying Lemma 2.4 to the previous family of confocal conics we obtain the following useful statement.

Lemma 2.7. *Let* α_1, α_2 *be two ellipses and* γ_1, γ_2 *be two hyperbolas from the same confocal family. Let* a, b, c, d *be their intersection points (see Fig. 5). Then the lines* (ac) *and* (bd) *touch a conic confocal to* $\alpha_1, \alpha_2, \gamma_1, \gamma_2$*.*

Now the classical Ivory theorem (see, for example, [12] and [8, sect. 30.6] follows directly from the Graves–Chasles theorem.

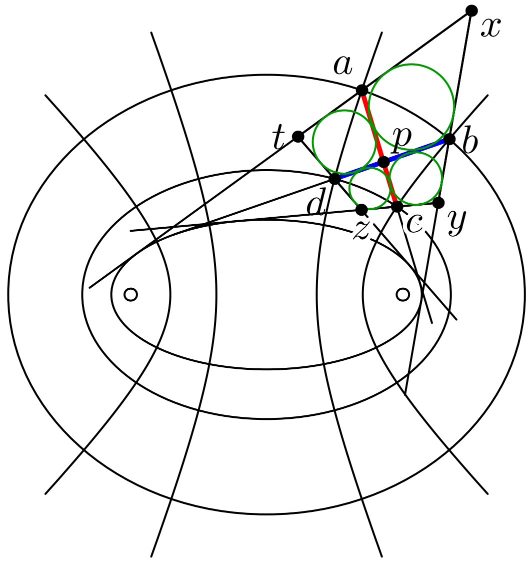


Figure 5: Proof of the Ivory theorem

Corollary 2.8 (Ivory theorem)**.** *Let α_1, α_2 be two ellipses and γ_1, γ_2 be two hyperbolas from the same confocal family. Let a, b, c, d be their intersection points (see Fig. 5). Then*

$$|ac| = |bd| \tag{2}$$

Proof. Due to Lemma 2.7 there exists a conic α from the confocal family with tangent lines (ac) and (bd). Let us draw four more lines tangent to α passing through the points a, b, c and d. They determine the intersection points x, y, z, t as in Fig. 5. The Graves–Chasles theorem implies that four obtained quadrilaterals shown in Fig. 5 as well as the big quadrilateral $(ztxy)$ are circumscribed.

Now identity (2) follows from the fact that the sum of lengths of two opposite edges of a circumscribed quadrilateral equals to the semiperimeter of the quadrilateral. Using the fact that the distances between the touching points on two exterior tangent lines common to two disjoint discs are equal it is easy to see that both $2|ac|$ and $2|bd|$ are equal to the sum of semiperimeters of quadrilaterals $(apdt)$, $(dpcz)$, $(axbp)$, and $(pbyc)$ minus the semiperimeter of $(ztxy)$. □

Let us give "global" definition of IC-net, which based Graves–Chasles theorem. Note that from Graves–Chasles theorem follows that points of intersection of $\{m_i, m_{i+1}\}$ and $\{\ell_j, \ell_{j+1}\}$ lie on a same conic from confocal family $\mathcal{A}$. Varying i and j we obtain that this conic is fixed. Using this observation, we can define IC-net in the following way.

Definition 3. Let α and α' be confocal conics. Let ℓ_i and m_j be a lines tangent to α and such that the point of intersections of $\{\ell_i, \ell_{i+1}\}$ and $\{m_j, m_{j+1}\}$ lie on α'. We call this lines as lines of IC-net and the points $f_{i,j} = \ell_i \cap m_j$ as vertices of IC-net.

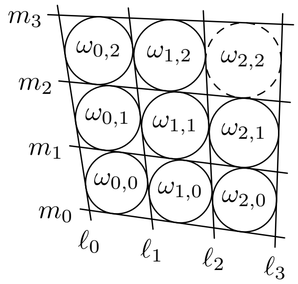


Figure 6: 3×3 incircles incidence theorem

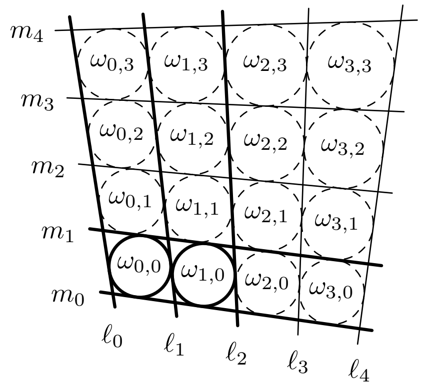


Figure 7: Construction of an IC-net from two circles and their five tangent lines

2.3 Construction of IC-nets

Construction of IC-nets is based on the following incidence theorem.

Theorem 2.9 (3×3 incircles incidence theorem). *Consider a quadrilateral which is cut in nine quadrilaterals by two pairs of lines ℓ_1, ℓ_2 and m_1, m_2 (see Fig. 6). Suppose all quadrilaterals except one at a corner are circumscribed. Then the ninth quadrilateral is also circumscribed.*

Proof. Consider the conic α which touches five lines $\ell_0, \ell_1, \ell_2, m_0, m_1$. Applying Corollary 2.6 several times we obtain that all lines in the figure are tangent to α. Denote by $f_{i,j}$ the intersection points $f_{i,j} = \ell_i \cap m_j$. Theorem 2.5 implies that the pairs $f_{2,3}$ and $f_{1,2}$, $f_{1,2}$ and $f_{2,1}$, $f_{2,1}$ and $f_{3,2}$ lie on conics confocal with α. Due to Lemma 2.7 $f_{2,3}$ and $f_{3,2}$ also lie on a conic confocal with α. Finally Theorem 2.5 implies that the quadrilateral formed by the lines ℓ_2, ℓ_3, m_2, m_3 is circumscribed. □

There is a natural way to construct an IC-net starting from two circles.

Corollary 2.10. *IC-nets considered up to Euclidean motions and homothety build a real four-dimensional family. An IC-net is uniquely determined by two neighboring circles $\omega_{0,0}, \omega_{1,0}$ and their tangent lines $\ell_0, \ell_1, \ell_2, m_0, m_1$ (see Fig. 7).*

Proof. Choose two non-intersecting circles $\omega_{0,0}$ and $\omega_{1,0}$ with tangent lines $\ell_0, \ell_1, \ell_2, m_0, m_1$ (see Fig. 7). Now the circles $\omega_{0,1}, \omega_{1,1}\omega_{2,0}$ are uniquely determined. Next the common tangent line m_2, and further, the circles $\omega_{0,2}, \omega_{1,2}, \omega_{2,1}$ are determined. They determine the tangent lines ℓ_3 and m_3. Finally the incribed circle $\omega_{2,2}$ exists due to the incidence theorem 2.9. Proceeding further this way one constructs the whole IC-net. □

2.4 Proof of Theorem 2.1

Actually we have proven already an essential part of Theorem 2.1.

(i),(ii) The corresponding properties were already proven for a 3×3 piece of an IC-net in Theorem 2.9. The global statement follows immediately.

(iii) follows directly from Theorem 2.5 and (i), (ii).

(iv) follows from the Ivory theorem (Corollary 2.8). Indeed, due to (ii) the points $f_{i-c,j}, f_{i,j-c}, f_{i+c,j}, f_{i,j+c}$ are the intersection points of two pairs of confocal ellipses and hyperbolas.

(v) It is well known (see, for example, [3]) that for any two lines tangent to a conic α the map from one to another generated by tangent lines to α is a projective map. Therefore it is preserves cross-ratios of points. The map

$$(f_{i_1,j_1}, f_{i_1,j_2}, f_{i_1,j_3}, f_{i_1,j_4}) \mapsto (f_{i_2,j_1}, f_{i_2,j_2}, f_{i_2,j_3}, f_{i_2,j_4})$$

is exactly of this type.

(vi) follows from the simple fact that the map between two conic of the same type from a confocal family generated by intersection by conics of other type is an affine map (see, for example, [8] or [10]). This fact follows immediately from the equations of confocal conics.

(vii),(viii) follow from the corresponding statements (ii),(v) of Theorem 3.1.

(ix),(x) On Fig. 1 is shown the IC-net, where lines ℓ_i and m_j coincide as a set.

Using the terms of Definition 3 it is easy to prove statements (x) and (ix) which follows directly from the previous and (ii) Note that bisectors of angles formed by lines $\{\ell_i, \ell_{i+1}\}$ or $\{m_j, m_{j+1}\}$ tangent to the conic α'. Denote them by ℓ'_i and m'_j respectively.

Note that point of intersection of lines ℓ'_i and ℓ'_{i+1} lie on fixed conic α'' dual to α with respect to α'. The same holds for intersection of lines m'_j and m'_{j+1}. Therefore, if one made a projective transform p which maps conics α' and α'' to confocal, the lines $p(\ell'_i)$ and $p(m'_j)$ will form a IC-net.

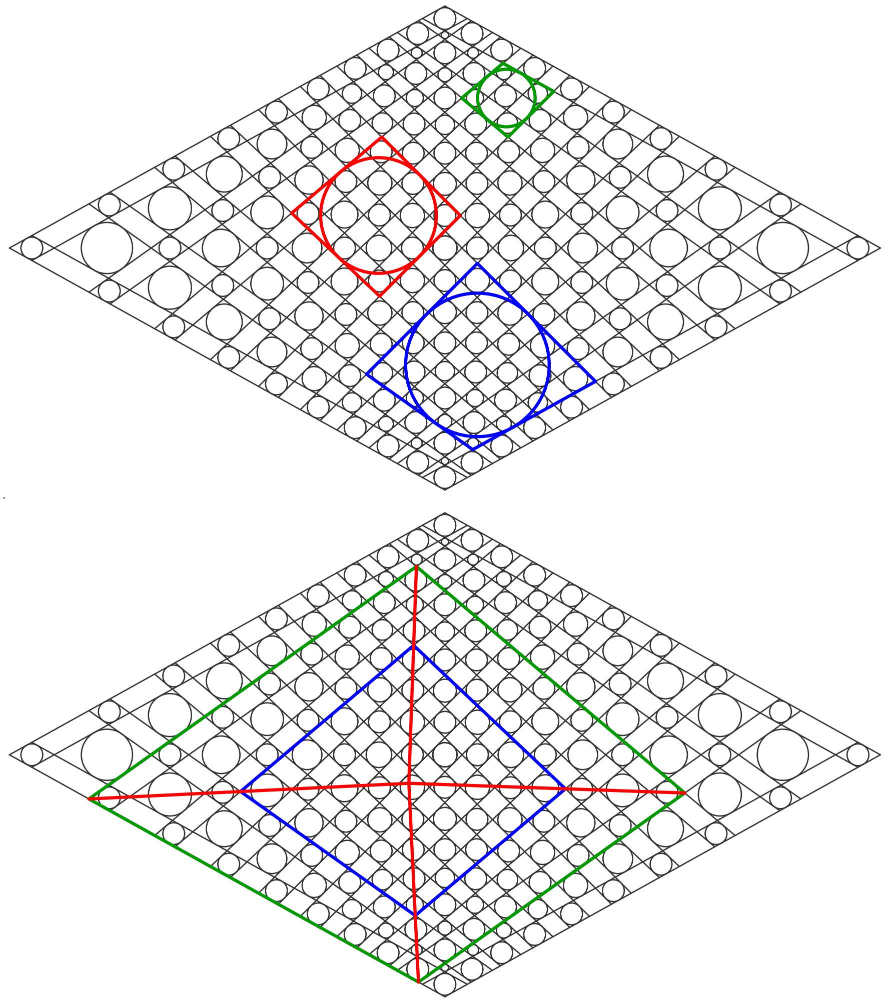


Figure 8: Checkerboard IC-nets: (top) circumscribed net-squares, (bottom) perspective net-squares

3 Checkerboard IC-net

3.1 Definition and geometric properties of checkerboard IC-nets

Definition 4. A checkerboard IC-net is a map $f : \mathbb{P} \to \mathbb{R}^2$ satisfying the following conditions:

1. For any integer i the points $\{f_{i,j} | j \in \mathbb{Z}\}$ lie on a straight line preserving the order, i.e the point $f_{i,j}$ lies between $f_{i,j-1}$ and $f_{i,j+1}$. The same holds for points $\{f_{i,j} | i \in \mathbb{Z}\}$. We call these lines the *lines of the checkerboard IC-net.*

2. For any integer i and j, such that $i + j$ is even the quadrilateral with verities $f_{i,j}$, $f_{i+1,j}$, $f_{i+1,j+1}$, $f_{i,j+1}$ is circumscribed.

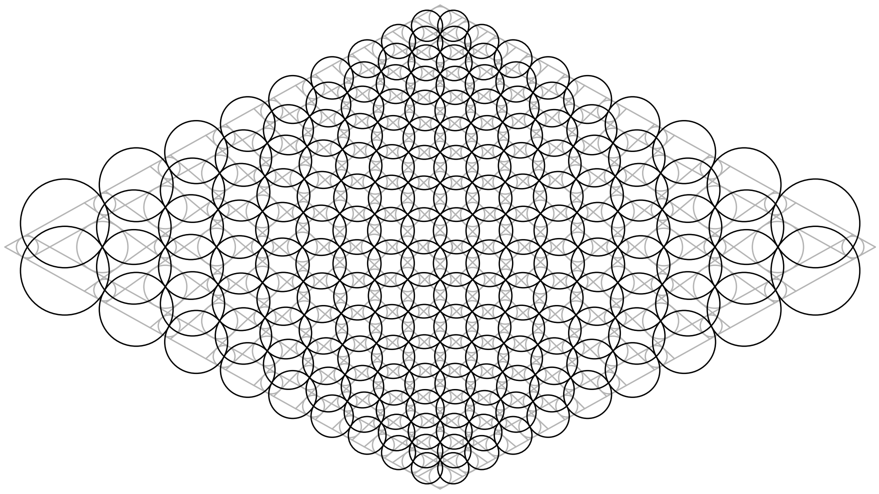


Figure 9: Checkerboard IC-nets as circular-conical nets

This class of nets with inscribed circles is some sense more natural then IC-nets. The reason is that all circles and lines of a checkerboard IC-net can be consistently oriented. This shows that this class of nets belongs to Laguerre geometry which studies oriented lines and circles that are in oriented contact (see, for example, [5]). We will use the Laguerre geometric description to prove some non-trivial incidence theorems which we have not found in the literature.

We call quadrilaterals $\square_{i,j}$ with vertices $f_{i,j}$, $f_{i+1,j}$, $f_{i+1,j+1}$, $f_{i,j+1}$ with even $i+j$ *unit net-squares* of checkerboard IC-net. The quadrilaterals $\square^c_{i,j}$ with vertices $f_{i,j}$, $f_{i+c,j}$, $f_{i+c,j+c}$, $f_{i,j+c}$ with even $i+j$ and odd c we call *net-squares.*

Theorem 3.1. *Let f be a checkerboard IC-net. Then the following properties hold:*

(i) All net-squares are circumscribed (Fig. 8 (top)).

(ii) Net-squares $\square^c_{i,j}$ and $\square^{c+2l}_{i-l,j-l}$, where l is odd, are perspective (Fig. 8 (bottom)).

(iii) The points $f_{i,j}$, where $i+j = \text{const}$ lie on a conic. The points $f_{i,j}$, where $i-j = \text{const}$ lie on a conic as well.

(iv) (Ivory-type theorem) We define the distance $d_C(\square_{a,b}, \square_{c,d})$ between two unit net-squares $\square_{a,b}$ and $\square_{c,d}$ of a checkerboard net as the distance between the tangent points on a common exterior tangent line to the circles $\omega_{a,b}$ and $\omega_{c,d}$ inscribed in $\square_{a,b}$ and $\square_{c,d}$ respectively. In case $a = c$ or $b = d$ these tangent lines are the lines of the checkerboard IC-net.

For any $(i,j) \in \mathbb{Z}^2$, with even $i+j$ and any integer even c one has

$$d_C(\square_{i-c,j}, \square_{i+c,j}) = d_C(\square_{i,j-c}, \square_{i,j+c}). \tag{3}$$

(v) Let $\omega_{i,j}$ be the inscribed circle of the unit net-square $\square_{i,j}$. Consider the cone in $\mathbb{R}^3$ intersecting the plane along $\omega_{i,j}$ at constant oriented angle (all the apexes $a_{i,j}$

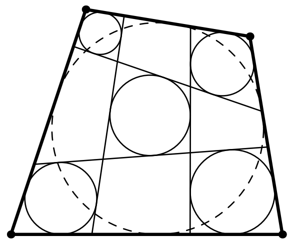


Figure 10: Six incirlces lemma

of these cones lie in one half-space). Then all the apexes $\{a_{i,j}|i+j = 4n, n \in \mathbb{Z}\}$ *lie on one-sheeted hyperboloid. The the apexes* $\{a_{i,j}|i + j = 4n + 2, n \in \mathbb{Z}\}$ *lie on one-sheeted hyperboloid as well.*

(vi) *The centers* $o_{i,j}$ *of the incircles of a checkerboard IC-net build a circle-conical net, i.e. a net that is simultaneously circular and conical (see [6]). Recall that circular nets are the nets with circular quadrilaterals* $(o_{i,j}o_{i+1,j+1}o_{i,j+2}o_{i-1,j+1})$ *and conical nets in plane are characterized by the condition that the sums of two opposite angles at a vertex are equal (and equal to* π*).*

We start with the following classical Lemma which can be found for example in [14].

Lemma 3.2. *Consider a quadrilateral which is cut in nine quadrilaterals by two pairs of lines (see Fig. 10). Suppose the center quadrilateral and all the corner quadrilaterals are circumscribed. Then the "big" quadrilateral is also circumscribed.*

An elementary proof of this lemma is based on the fact that the distances between the touching points on two exterior tangent lines common to two disjoint discs are equal. Using this fact one can show that the differences of the sums of the lengths of the opposite sides of the central "small" and "big" quadrilaterals are equal. This implies that they are simultaneously circumscribed.

Proof. of Theorem 3.1

(i) We will prove this by induction on c. From the definition we obtain the claim for $c = 1$. Suppose the circumcrability is known for all net-squares of size c. Let us prove it for net-squares of size $c + 2$. Applying Lemma 3.2 for the unit net-squares $\Box_{i,j}$, $\Box_{i,j+c+1}$, $\Box_{i+c+1,j+c+1}$, $\Box_{i+c+1,j}$ and the net-square $\Box^c_{i+1,j+1}$ we get that the net-square $\Box^{c+2}_{i,j}$ is also circumscribed.

(ii) is a two-dimensional version of the property (ii) of Theorem 4.1.

(iii) We will prove the claim for vertices $f_{i,j}$ with $i+j = 0$, for other cases the proof is the same. For that we will show that any six successive points lie on a some conic. Without loss of generality, we assume that $i = 1$. By the Pascal theorem it is enough to show that the following three points of intersection of lines $(f_{1,-1}f_{2,-2}) \cap (f_{4,-4}f_{5,-5})$, $(f_{2,-2}f_{3,-3}) \cap (f_{5,-5}f_{6,-6})$, $(f_{3,-3}f_{4,-4}) \cap (f_{6,-6}f_{1,-1})$ are collinear.

The point $f_{1,-1}$ is the center of positive homothety of incircles of the net-squares $\square_{1,-1}$ and $\square^3_{1,-1}$. The point $f_{2,-2}$ is a center of negative homothety of incircles of the net-squares $\square_{1,-1}$ and $\square^3_{2,-2}$. Therefore, by the Monge theorem the line $(f_{1,-1}f_{2,-2})$ passes through the center of negative homothety of incircles of the net-squares $\square^3_{2,-2}$ and $\square^3_{1,-1}$. Analogously the line $(f_{4,-4}f_{5,-5})$ passed through this center. We obtain that the point $(f_{1,-1}f_{2,-2}) \cap (f_{4,-4}f_{5,-5})$ is the center of negative homothety of the incircles of $\square^3_{2,-2}$ and $\square^3_{1,-1}$.

Using the same argument we can prove that $f_{2,-2}f_{3,-3} \cap f_{5,-5}f_{6,-6}$ is the center of negative homothety of incircles of $\square^3_{2,-2}$ and $\square^3_{3,-3}$. The point $(f_{3,-3}f_{4,-4}) \cap (f_{6,-6}f_{1,-1})$ is the center of positive homothety of $\square^3_{1,-1}$ and $\square^3_{3,-3}$. Applying the Monge theorem again we obtain that these three centers of homotheties lie on a one line.

(iv) Since the net-square $\square^{c+1}_{i,j}$ is circumscribed the sums of the lengths of its opposite sides are equal. The sum of the lengths of two opposite sides of $\square^{c+1}_{i,j}$ is equal to $d_C(\square_{i,j}, \square_{i+c,j}) + d_C(\square_{i,j+c}, \square_{i+c,j+c})$ plus the sum of the lengths of the intervals from the corners of $\square^{c+1}_{i,j}$ to the touching points with the inscribed circles of corresponding corner unit net-squares. We obtain

$$d_C(\square_{i,j}, \square_{i+c,j}) + d_C(\square_{i,j+c}, \square_{i+c,j+c}) = d_C(\square_{i,j}, \square_{i,j+c}) + d_C(\square_{i+c,j}, \square_{i+c,j+c}).$$

Applying this equality to $\square^{c+1}_{i-c,j-c}$, $\square^{c+1}_{i-c,j}$, $\square^{c+1}_{i,j-c}$, $\square^{c+1}_{i,j}$ and $\square^{2c+1}_{i-c,j-c}$ we get

$$\begin{aligned}
2d_C(\square_{i-c,j}, \square_{i+c,j}) &= 2d_C(\square_{i-c,j}, \square_{i,j}) + 2d_C(\square_{i,j}, \square_{i+c,j}) = \\
&= d_C(\square_{i-c,j-c}, \square_{i-c,j}) + d_C(\square_{i,j-c}, \square_{i,j}) - d_C(\square_{i-c,j-c}, \square_{i,j-c}) + \\
&\quad + d_C(\square_{i-c,j}, \square_{i-c,j+c}) + d_C(\square_{i,j}, \square_{i,j+c}) - d_C(\square_{i-c,j}, \square_{i,j}) + \\
&\quad + d_C(\square_{i,j-c}, \square_{i,j}) + d_C(\square_{i+c,j-c}, \square_{i+c,j}) - d_C(\square_{i,j-c}, \square_{i+c,j-c}) + \\
&\quad + d_C(\square_{i,j}, \square_{i,j+c}) + d_C(\square_{i+c,j}, \square_{i+c,j+c}) - d_C(\square_{i,j}, \square_{i+c,j}) = \\
&= d_C(\square_{i-c,j-c}, \square_{i-c,j+c}) + d_C(\square_{i+c,j-c}, \square_{i+c,j+c}) - \\
&\quad - d_C(\square_{i-c,j-c}, \square_{i+c,j-c}) - d_C(\square_{i-c,j+c}, \square_{i+c,j+c}) + 2d_C(\square_{i,j-c}, \square_{i,j+c}) = \\
&= 2d_C(\square_{i,j-c}, \square_{i,j+c}). \qquad (4)
\end{aligned}$$

(v) The apexes of the cones of unit net-squares with fixed i (or fixed j) are collinear. The lines of different types (with fixed i or fixed j) intersect each other. Therefore they are asymptotic lines (of two different families) of an one-sheeted hyperboloid.

(vi) Both angle conditions of circularity and of conicality follow immediately.

□

3.2 Construction of checkerboard IC-nets

Theorem 3.3 (checkerboard incircles incidence theorem)**.** *Consider a quadrilateral which is cut by two sets of four lines in* 25 *quadrilaterals. Color the quadrilaterals in a checkerboard pattern with black quadrilaterals at the corners. Assume*

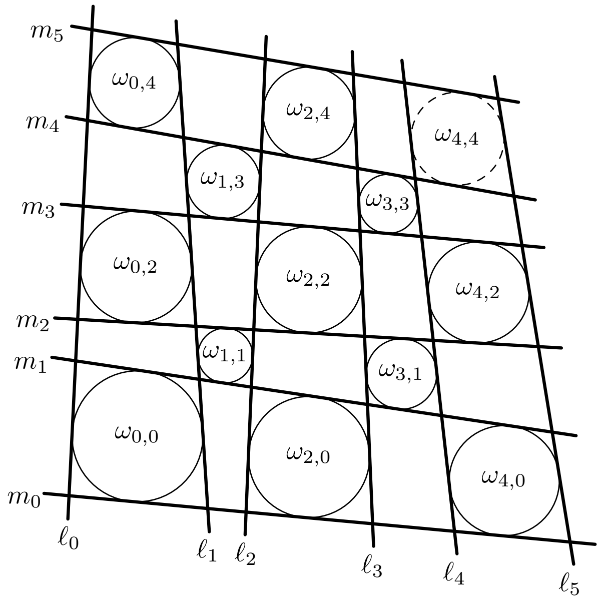


Figure 11: Checkerboard incircles incidence theorem

that all black quadrilaterals except one at a corner are circumscribed. The the the last black quadrilateral at the corner (thirteenth quadrilateral) is also circumscribed (Figure 11).

Before we prove this theorem let us make an important comment. As we have already pointed out in Section 3.1 checkerboard IC-nets can be oriented in such a way that their circles and lines are in oriented contact. They can be naturally described in frames of Laguerre geometry. Let us briefly introduce the cyclographic model of Laguerre geometry in the plane (see [5]).

In this model the space of oriented circles $C = \{x \in \mathbb{R}^2 | |x - c|^2 = r^2\}$ is in one-to-one correspondence with the points $a = (c, r)$ of the Minkowski space $\mathbb{R}^{2,1}$. They can be seen as the apexes of the cones of revolution intersecting the plane $\mathbb{R}^2 \subset \mathbb{R}^{2,1}$ at the angle $\pi/4$ along the circles C. The oriented lines $\ell \in \mathbb{R}^2$ are modelled as oriented planes $L \subset \mathbb{R}^{2,1}$ intersecting the plane $\mathbb{R}^2$ along the lines ℓ at the angle $\pi/4$.

An oriented circle C is n oriented contact with an oriented line ℓ if and only if $a \in L$. Two oriented circles C_1, $C_2 \subset \mathbb{R}^2$ are in oriented contact if and only if their representatives in the Minkowski space a_1, $a_2 \in \mathbb{R}^{2,1}$ differ by an isotropic vector $|a_1 - a_2| = 0$.

In this model a one-parameter family of circles that are in oriented contact to two oriented lines is represented by a straight line in $\mathbb{R}^{2,1}$. The points of this line are the apexes of the corresponding cones.

Proof. The circles $\omega_{i,j}$ are inscribed in $\square_{i,j}$. Let us orient the circles $\omega_{i,j}$ with even i, j positively, the circles $\omega_{i,j}$ with odd i, j negatively, and the tangent lines ℓ, m so that they are in oriented contact to these circles. Due to Lemma 10 the net-square $\square^3_{1,1}$ is also circumscribed, denote its inscribed circle as $\omega^3_{1,1}$ and orient it so that it is in oriented contact to the lines.

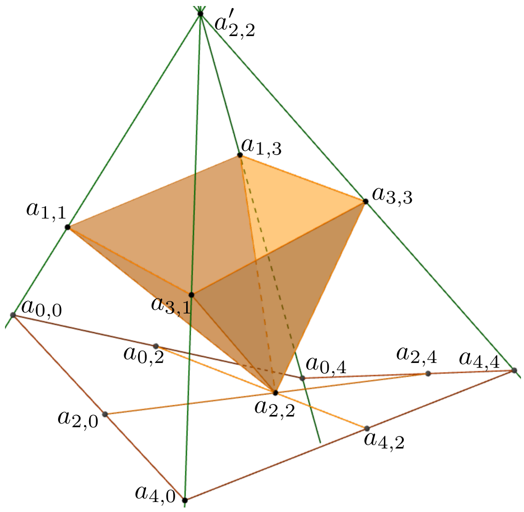


Figure 12: Projective octahedron incidence theorem

Consider the Laguerre geometry of this pattern in the cyclographic model described above. Let $a_{i,j}$ be the apex of the cone which intersect the plane at the angle $\pi/4$ along $\omega_{i,j}$. Denote by $a'_{2,2}$ the apex on the cone corresponding to the circle $\omega^3_{1,1}$. The orientation described above implies that the points $a_{1,1}$, $a_{1,3}$, $a_{3,1}$, $a_{3,3}$, $a'_{2,2}$ lie in one halfspace of $\mathbb{R}^{2,1}$ (let us say, positive third component r), and the points $a_{i,j}$ with even i, j in the other halfspace (negative r).

Apexes a's are collinear if and only if the corresponding circles share two common tangent lines. The following triples of points are collinear $\{a'_{2,2}, a_{1,1}, a_{0,0}\}$, $\{a'_{2,2}, a_{1,3}, a_{0,4}\}$, $\{a'_{2,2}, a_{3,1}, a_{4,0}\}$, $\{a_{2,0}, a_{2,2}, a_{2,4}\}$, and $\{a_{0,2}, a_{2,2}, a_{4,2}\}$. Moreover the apexes are coplanar if and only if the corresponding circles share a common tangent line. The following quintuples of points are coplanar $\{a_{0,2}, a_{2,2}, a_{4,2}, a_{1,1}, a_{3,1}\}$, $\{a_{0,2}, a_{2,2}, a_{4,2}, a_{1,3}, a_{3,3}\}$, $\{a_{2,0}, a_{2,2}, a_{2,4}, a_{1,1}, a_{1,3}\}$, $\{a_{2,0}, a_{2,2}, a_{2,4}, a_{3,1}, a_{3,3}\}$.

We obtain a projective octahedron $a_{2,2}, a_{,1}, a_{3,1}, a_{3,3}, a_{1,3}, a'_{2,2}$ as in Fig. 3.2. The intersection line of the face planes $(a_{1,1}a_{1,3}a_{2,2})$ and $(a_{3,1}a_{3,3}a_{2,2})$ intersects the planes $(a_{1,1}a_{3,1}a'_{2,2})$ and $(a_{3,3}a_{1,3}a'_{2,2})$ at the points $a_{2,0}$ and $a_{2,4}$ respectively. Analogously,

$$
\begin{aligned}
a_{0,2} &= (a_{1,1}a_{3,1}a_{2,2}) \cap (a_{1,3}a_{3,3}a_{2,2}) \cap (a_{1,1}a_{1,3}a'_{2,2}) \\
a_{4,2} &= (a_{1,1}a_{3,1}a_{2,2}) \cap (a_{1,3}a_{3,3}a_{2,2}) \cap (a_{3,1}a_{3,3}a'_{2,2})
\end{aligned} \tag{5}
$$

The rest of the proof follows from Theorem 3.4. □

Theorem 3.4 (Projective octahedron incidence theorem)**.** *Consider a projective octahedron as in Fig. 3.2 and define the points $a_{0,2}$, $a_{2,0}$, $a_{4,2}$, $a_{2,4}$ as the intersection points of the corresponding face planes like (5). Chose an arbitrary point $a_{0,0} \in (a_{1,1}a'_{2,2})$, determine $a_{4,0} := (a_{0,0}a_{2,0}) \cap (a_{3,1}a'_{2,2})$ and $a_{0,4} := (a_{0,0}a_{0,2}) \cap (a_{1,3}a'_{2,2})$. Then the lines $(a_{0,4}a_{2,4})$ and $(a_{4,0}a_{4,2})$ intersect the line $(a_{3,3}a'_{2,2})$ in a common point (which we denote by $a_{4,4}$.*

Proof. Since the claim is projective we may simplify the representation by mapping the lines $(a_{2,0}a_{2,4})$ and $(a_{0,2}a_{4,2})$ to the infinity plane. Then point $a_{2,2}$ also lies in

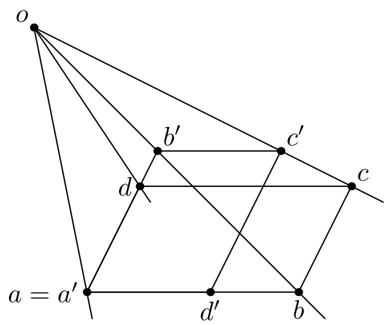


Figure 13: Proof of Lemma 3.5 from Pappus theorem

this plane. The planes $(a_{1,1}a_{3,1}a_{2,2})$ and $(a_{1,3}a_{3,3}a_{2,2})$ become two parallel planes, the planes $(a_{1,1}a_{1,3}a_{2,2})$ and $(a_{3,1}a_{3,3}a_{2,2})$ are parallel as well.

The straight line $(a_{0,0}a_{4,0})$ is the intersection line the planes $(a_{1,1}a_{3,1}a'_{2,2})$ and $(a_{0,0}a_{2,0}a_{2,2})$. After our normalization the later becomes the plane parallel to $(a_{1,1}a_{1,3}a_{2,2})$ (and $(a_{3,1}a_{3,3}a_{2,2})$) and passing through $a_{0,0}$. Finally we obtain the lines $(a_{0,0}a_{4,0})$, $(a_{0,4}a_{2,4})$ and $(a_{0,0}a_{0,4})$, $(a_{4,0}a_{4,2})$ as the intersections of the corresponding face planes at $a'_{2,2}$ with the planes parallel to $(a_{1,1}a_{1,3}a_{2,2})$ and $(a_{1,1}a_{1,3}a_{2,2})$ respectively.

Now projecting the whole geometry to a plane (transversal to the line $(a_{2,2}a_{1,1})$ etc.) we obtain the incidence statement from Lemma 3.5 in plane geometry. □

Lemma 3.5. *Let $(abcd)$ be a parallelogram and o be a point in plane which does not lie on the lines of the sides of the parallelogram. Let a' be a point on (oa). Then there exists a unique parallelogram $(a'b'c'd')$ such that its lines $(a'b')$, $(b'c')$, $(c'd')$, $(d'a')$ pass through the point o and the non-corresponding sides are parallel to the sides of the original parallelogram: $(a'd') \parallel (ab)$ and $(c'd') \parallel (bc)$.*

Proof. Without loss of generality one can assume $a' = a$. After that the claim is just the Pappus theorem for points b, c, d, b', c', d', o and two points at infinity. □

The dual version of this Lemma formulated for conics can be found in [1].

Corollary 3.6. *Checkerboard IC-nets considered up to Euclidean motions and homothety build a real eight-dimensional family. A checkerboard IC-net is uniquely determined by five neighboring circles $\omega_{0,0}$, $\omega_{2,0}$, $\omega_{0,2}$, $\omega_{2,2}$, $\omega_{1,1}$ and circle $\omega_{3,3}$ (see Fig. 14).*

Proof. For constructing of a checkerboard IC-net we start with a circle $\omega_{1,1}$ and its four tangents ℓ_1, ℓ_2, m_1 and m_2. Then we inscribe circles $\omega_{0,0}$, $\omega_{0,2}$, $\omega_{2,2}$ and $\omega_{2,0}$ in the corners (see Figure 14). The four common tangents lines ℓ_0, ℓ_3, m_0, and m_3 are uniquely determined. Further, the circles $\omega_{1,3}$ and $\omega_{3,1}$ are uniquely determined. By choosing the circle $\omega_{3,3}$ we have one degree of freedom. Now, the whole net is fixed. Indeed, we consequently determine the lines ℓ_4 and m_4, then the circles $\omega_{4,0}$, $\omega_{4,2}$, $\omega_{0,4}$, $\omega_{2,4}$ and lines ℓ_5 and m_5. The existence of the circle $\omega_{4,4}$ follows from the incidence theorem 3.3. Applying this theorem again and again one generates the whole checkerboard IC-net. □

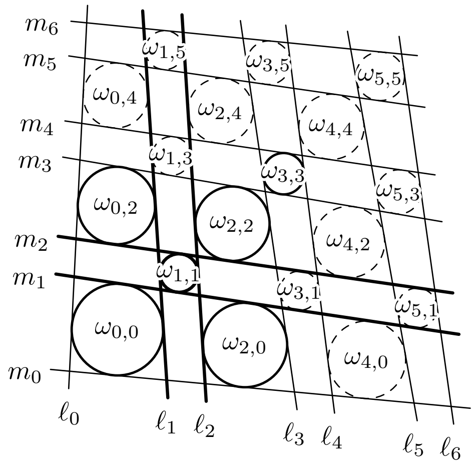


Figure 14: Construction of a checkerboard IC-net

3.3 Checkerboard confocal IC-net

IC-nets can be considered as special checkerboard IC-nets. Indeed, if for every second line and every second column of a checkerboard IC-net the incircles degenerate to points, then the lines of the net merge in pairs, all non-circumscribed quadrilaterals disappear, and one obtains an IC-net.

There is however an interesting class of IC-nets, which lies between the two we have considered. These are special checkerboard IC-nets related to confocal conics.

Definition 5. We call a checkerboard IC-net *confocal* if all lines of it are tangent to a conic.

This class is a natural generalization of IC-nets introduced in Section 2. However in contrast to IC-nets here all circles and lines can be oriented so that and the corresponding circles and lines are in oriented contact. This class can be studied in frames of Laguerre geometry.

We formulate the most important geometric property of checkerboard confocal IC-nets which can be proven exactly in the same way as the corresponding theorems in Sections 2.1 and 3.1.

Theorem 3.7. *Let f be a checkerboard confocal IC-net all lines of which are tangent to a conic α. Then the points $f_{i,j}$, where $i+j=\text{const}$ lie on a conic confocal with α. As well the points $f_{i,j}$, where $i-j=\text{const}$ lie on a conic confocal with α.*

Let us give "global" definition of Checkerboard confocal IC-net similar to Definition 3 of IC-net.

Definition 6. Let α, α' and α'' be confocal conics. Let ℓ_i and m_i be a lines tangent to α and such that for odd i the points $\ell_i \cap \ell_{i+1}$ and $m_i \cap m_{i+1}$ lie on α' and for even i on α''. We call this lines as lines of Checkerboard confocal IC-net and the points $f_{i,j} = \ell_i \cap m_j$ as vertices of IC-net.

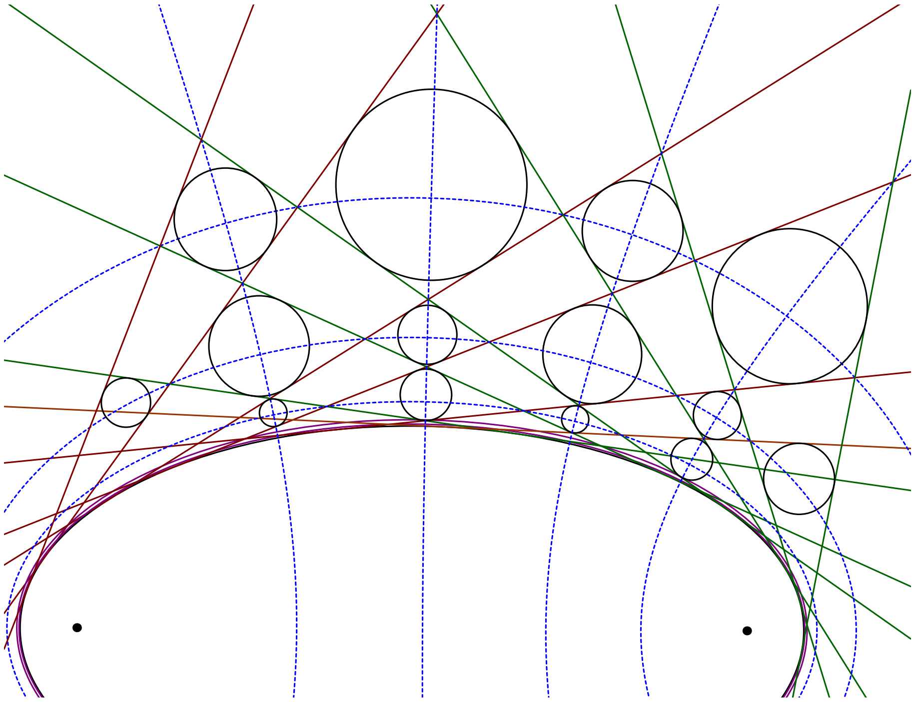


Figure 15: To a definition of checkerboard confocal IC-nets

4 Checkerboard inspherical nets in $\mathbb{R}^3$

4.1 Definition and geometric properties of checkerboard IS-nets

In this section we consider a natural multi-dimensional version of checkerboard IS-nets. We consider maps of the integer grid $f : \mathbb{Z}^n \to \mathbb{R}^n$ or of a quadrangle $\mathbb{P} \subset \mathbb{Z}^n$ of full dimension. Although all results are valid for any dimension to simplify the notations we formulate some of them in dimension three. Let us denote $\cube^c_{i,j,k}$ the cube with the vertices $f_{i,j,k}$,$f_{i+c,j,k}$, $f_{i,j+c,k}$, $f_{i,j,k+c}$,$f_{i+c,j+c,k}$, $f_{i,j+c,k+c}$, $f_{i+c,j,k+c}$,$f_{i+c,j+c,k+c}$.

We call $\cube^c_{i,j,k}$ a *net cube* if i, j, k are all even or all odd and c is odd. *Unit net-cubes* are the net-cubes $\cube_{i,j,k} = \cube^1_{i,j,k}$.

Definition 7. A checkerboard IS-net (inscribed spherical net) is a map $f : \mathbb{P} \to \mathbb{R}^3$ satisfying the following conditions:

(i) For any integer i the points $\{f_{i,j,k} | k \in \mathbb{Z}\}$ lie on a straight line preserving the order, i.e the point $f_{i,j,k}$ lies between $f_{i,j,k-1}$ and $f_{i,j,k+1}$. The same holds for points $\{f_{i,j,k} | i \in \mathbb{Z}\}$ and $\{f_{i,j,k} | j \in \mathbb{Z}\}$. We call these lines the *lines of the checkerboard IS-net.*

(ii) The unit net-cubes $\cube_{i,j,k}$ are circumscribed cubes.

Let us denote by $\ell_{j,k}$ the line of the checkerboard IS-net that contains the vertexes $\{f_{i,j,k} | \forall i\}$, similarly denote the lines of the other two families by $m_{i,k} \supset \{f_{i,j,k} | \forall j\}$

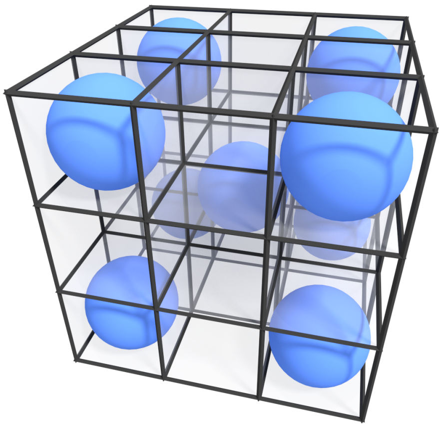


Figure 16: Body-centered cubic of IS-net

and $n_{i,j} \supset \{f_{i,j,k}|\forall k\}$. The planes of the checkerboard IS-net we denote by $L_i \supset \{f_{i,j,k}|\forall j,k\}$, $M_j \supset \{f_{i,j,k}|\forall i,k\}$, and $N_k \supset \{f_{i,j,k}|\forall i,j\}$.

Theorem 4.1. *(i) All net-cubes of an IS-net are circumscribed.*

(ii) The net-cubes $\square^c_{i,j,k}$ and $\square^{c+4s+2}_{i-2s-1,j-2s-1,k-2s-1}$ are perspective.

(iii) All net-cubes are projective cubes, i.e. projective images of the standard cube.

(iv) The lines $\ell_{j,k}$, where $j+k = \mathrm{const}$ lie on hyperboloid of one sheet. The same holds for the lines $\ell_{j,k}$ with $j-k = \mathrm{const}$ and for the corresponding lines $m_{i,k}$ and $n_{i,j}$.

4.2 Construction of checkerboard IS-nets

A polytopes combinatorial equivalent to the cube we call *combinatorial cube.*

Theorem 4.2 (9 inspheres incidence theorem)**.** *Suppose combinatorial cube $\square$ in $\mathbb{R}^3$ cut by 6 planes on 27 combinatorial cubes. Suppose the central and seven of the corner cells are circumscribed. Then the last corner cell is also circumscribed.*

Proof. As in this section denote the vertices of this cell devision $f_{i,j,k}$. By $\square_{i,j,k}$ denote the cell with vertices with indexes i or $i+1$, j or $j+1$, k or $k+1$. The centers of inscribed spheres (if they exist) of denote by $o_{i,j,k}$. By L^x_i denote the planes. Suppose eighth corner cell in the statement is the cell $\square_{2,2,2}$.

Consider projective map which maps $o_{1,1,1}$ to itself and maps points $f_{1,1,1}$, $f_{2,1,1}$, $f_{1,2,1}$, and $f_{1,1,2}$ to points $o_{0,0,0}$, $o_{2,0,0}$, $o_{0,2,0}$, and $o_{0,0,2}$ correspondingly. Note that this map should maps the lines passing through $o_{1,1,1}$ to themselves (because it preserve four of them, which are in general position).

The plane L^x_1 is mapped to the plane passing through $o_{0,0,0}$, $o_{0,2,0}$, and $o_{0,0,2}$. Therefore the point $f_{1,2,2}$ mapped to intersection of the line $f_{1,2,2}o_{1,1,1}$ and the bisector of planes L^x_0 and L^x_1, the point $o_{0,2,2}$. Analogously we show that the images

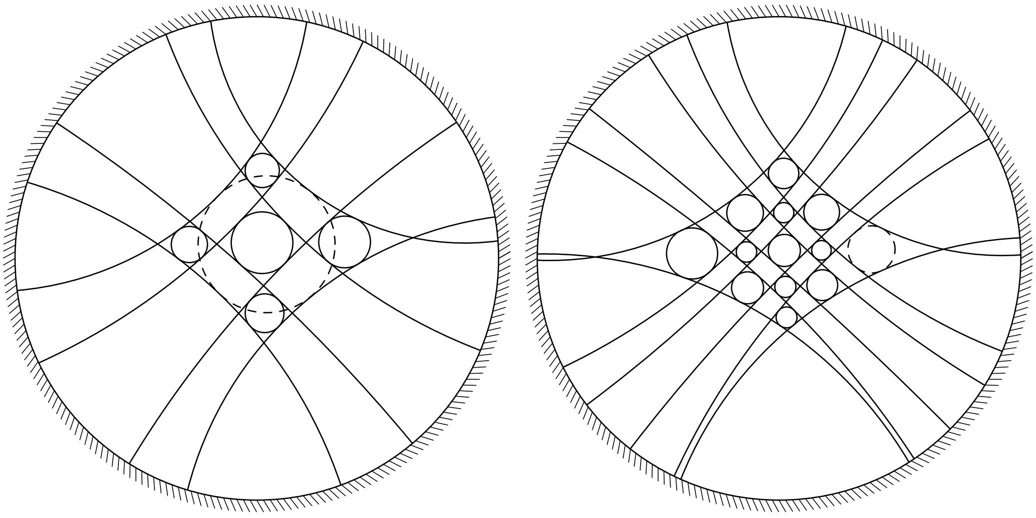


Figure 17: Hyperbolic six incircles lemma and checkerboard incircles incidence theorem

of vertices of $\square_{1,1,1}$ map to the centers of inscribed spheres of corner cells. So the point $o_{2,2,2}$ exists as intersection of three bisectors planes between pairs of planes L^x_2 and L^x_3, L^y_2 and L^y_3, L^z_2 and L^z_3 which lies on the line $o_{1,1,1}f_{2,2,2}$. That means that this point in the same distances from all sides of $\square_{2,2,2}$, therefore the cell is circumscribed. $\square$

5 Incircular nets in hyperbolic plane

Almost all claims from the previous sections are valid in the hyperbolic and spherical geometries.

The main statements from Section 4 also hold.

6 Appendix. An elementary proof of the Graves-Chasles theorem

Proof. Let us show that (ii) implies (i). We assume that α is an ellipse with foci f_1 and f_2. The case of hyperbola can be proven in the same way. Let α_1 be the ellipse passing through the points a and c, and $b_1 := (af_1) \cap (bf_2)$, $d_1 := (af_2) \cap (bf_1)$. Let f'_1 be a reflection of f_1 in line (ab). Note that $|f'_1 f_2|$ equals to length l of the the big axis of the ellipse α. We denote by l_1 the length of the big axis of α_1.

Since a and c lie on the ellipse α_1, we have $|f_1 a| + |f_2 a| = |f_1 b| + |f_2 b|$. Therefore the quadrilateral (ab_1cd_1) is circumscribed. Denote the center of its circle by i. From Poncelet's isogonal lemma (see, for example, Theorem 1.4 in [3]) it follows that $\angle b_1ab = \angle dad_1$ and $\angle b_1cb = \angle dcd_1$. So, it is sufficient to show that the

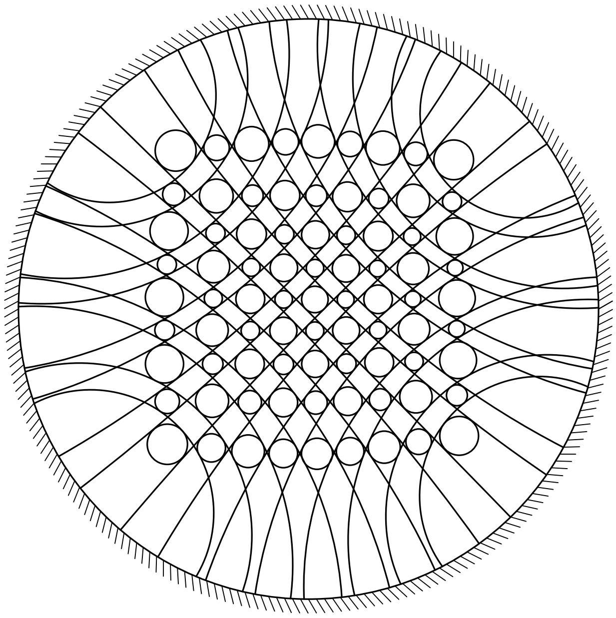


Figure 18: Hyperbolic checkerboard IC-net

distances from i to (ad) and (cd) are equal. We observe that

$$\frac{d(i,ad)}{d(i,ad_1)} = \frac{\sin\angle iad}{\sin\angle iad_1} = \frac{\cos\frac{1}{2}(\pi - \angle bad)}{\cos\frac{1}{2}(\pi - \angle b_1ad_1)} = \frac{\cos\frac{1}{2}\angle f_1'af_2}{\cos\frac{1}{2}\angle f_1af_2}.$$

Further $2(\cos\frac{1}{2}\angle f_1'af_2)^2 = \cos\angle f_1'af_2 + 1$ holds, and from the cosine formula we obtain $\cos\angle f_1'af_2 = \frac{|f_1a|^2+|f_2a|^2-l^2}{2|f_1a|\cdot|f_2a|}$. And we get $\cos\angle f_1'af_2/2 = \sqrt{\frac{l_1^2-l^2}{2|f_1a|\cdot|f_2a|}}$. An analogous computation gives $2(\cos\frac{1}{2}\angle f_1af_2)^2 = \cos\angle f_1af_2 + 1$, $\cos\angle f_1af_2 = \frac{|f_1a|^2+|f_2a|^2-|f_1f_2|^2}{2|f_1a|\cdot|f_2a|}$, $\cos\frac{1}{2}\angle f_1af_2 = \sqrt{\frac{l_1^2-|f_1f_2|^2}{2|f_1a|\cdot|f_2a|}}$. Therefore

$$\frac{d(i,ad)}{d(i,ad_1)} = \frac{\cos\frac{1}{2}\angle f_1'af_2}{\cos\frac{1}{2}\angle f_1af_2} = \sqrt{\frac{l_1^2-l^2}{l_1^2-|f_1f_2|^2}}.$$

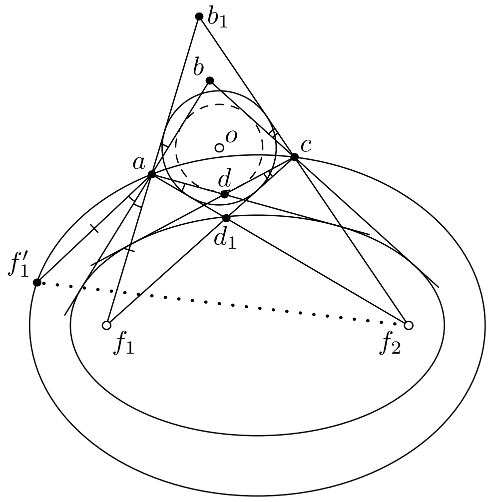
We see that the ratio $\frac{d(i,ad)}{d(i,ad_1)}$ is independent of the point a. Hence

$$\frac{d(i,ad)}{d(i,ad_1)} = \frac{d(i,bd)}{d(i,bd_1)},$$

and finally $d(i,ad) = d(i,cd)$, since $d(i,ad_1) = d(i,cd_1)$.

Let us show now that (i) implies (ii).

If c does not lie on the ellipse with foci f_1 and f_2 passing through a we can choose another point c' on (bc) such that it is and define d' as the point of intersection of (ad) with the tangent line from c' to α. The quadrilateral $abc'd'$ is circumscribed. But on the other hand, the incircles of $(abc'd')$ and $(abcd)$ coincide, and (cd) and $(c'd')$ are the common interior tangent lines of α and this incircle. They should coincide, thus $c_1 = c$. The equivalence (i) $\Leftrightarrow$ (iii) can be shown in the same way. $\square$



References

[1] V. E. Adler. Some incidence theorems and integrable discrete equations. *Discrete & Computational Geometry*, 36(3):489–498, 2006.

[2] A. V. Akopyan and A. A. Zaslavsky. Different views on the isogonal conjugation. *Matematicheskoe Prosveschenie. Tret'ya Seriya.*, 11:61–78, 2007.

[3] A. V. Akopyan and A. A. Zaslavsky. *Geometry of conics*, volume 26 of *Mathematical World*. American Mathematical Society, Providence, RI, 2007. Translated from the 2007 Russian original by Alex Martsinkovsky.

[4] M. Berger. *Geometry*. Springer Verl., 2009.

[5] W. Blaschke and G. Thomsen. *Vorlesungen über Differentialgeometrie und Geometrische Grundlagen von Einsteins Relativitätstheorie: III: Differentialgeometrie der Kreise und Kugeln*. Springer Berlin Heidelberg, 1929.

[6] A. I. Bobenko and Y. B. Suris. *Discrete differential geometry: Integrable structure*, volume 98. American Mathematical Soc., 2008.

[7] G. Darboux. *Principes de géométrie analytique*. Paris, 1917.

[8] D. Fuchs and S. Tabachnikov. *Mathematical omnibus: thirty lectures on classic mathematics*. Springer, 2007.

[9] P. Griffiths and J. Harris. A Poncelet theorem in space. *Commentarii Mathematici Helvetici*, 52(1):145–160, 1977.

[10] M. Levi and S. Tabachnikov. The Poncelet grid and billiards in ellipses. *The American Mathematical Monthly*, 114(10):895–908, 2007.

[11] R. E. Schwartz. The Poncelet grid. *Advances in Geometry*, 7(2):157–175, 2007.

[12] H. Stachel and J. Wallner. Ivory's theorem in hyperbolic spaces. *Siberian Mathematical Journal*, 45(4):785–794, 2004.

[13] E. Tsukerman. Discrete conics as distinguished projective images of regular polygons. *Discrete Comput. Geom.*, 53(4):691–702, 2015.

[14] I. Yaglom. *Complex numbers in geometry.* New York: Academic Press, 1968.