CLUSTER ALGEBRAS AND COXETER GROUPS

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Abstract. Coxeter groups are classical objects appearing in connection with symmetry groups of regular polytopes and various tessellations. Cluster algebras were introduced by Fomin and Zelevinsky in 2002 and gained a growing wave of interest due to numerous relations to other fields in mathematics and theoretical physics. We will discuss a number of connections between cluster algebras and Coxeter groups.

Contents

Introduction

Coxeter groups are groups of symmetries of regular polytopes, various tilings and other beautiful pictures, like ones you can see in a kaleidoscope. They are named after H. S M. Coxeter who classified finite and affine Coxeter groups in 1934 [Cox].

Cluster algebras were introduced by Fomin and Zelevinsky in 2002 [FZ1] in connection with totally positive matrices. Since then cluster algebra structures were found in growing number of contexts, including algebraic geometry, representation theory, discrete dynamical systems, Teichmüller theory, Poisson geometry and many others.

In these notes we will discuss a number of connections between cluster algebras and Coxeter groups.

The structure of the notes is as follows:

- Lecture 1 is devoted to background related to Coxeter groups.
- Lecture 2 introduces cluster algebras, it is a fresh start completely independent of Lecture 1.
- Lecture 3 discusses a number of connections between cluster algebras and Coxeter groups.

Material in the blue boxes is complementary, it can be easily ommited.

In the green boxes we collect open problems related to the discussion.

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1. Lecture 1: Geometry of Coxeter groups

In this section we discuss geometric aspects of Coxeter groups. For the reference one can take any of the books [B], [D], [Hum].

1.1. **Coxeter groups as reflection groups.**

Definition 1.1 (Coxeter group)**.**

- A group G is a *Coxeter group* if it has a presentation
$$G = \langle s_, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle,$$
where $m_{ij} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Here, $m_{ij} = \infty$ means that $s_i s_j$ is an element of infinite order (so, there is no corresponding relation).
- The generators $s_1, \dots, s_n$ and any elements conjugated to them in G are called *reflections*.

The collocation (G, S), $S = \{s_1, \dots, s_n\}$ in the definition above is called a *Coxeter system*.

Coxeter groups arise as symmetry groups of regular polytopes, bathroom floor decorations and various tilings like the one in Fig. 1.

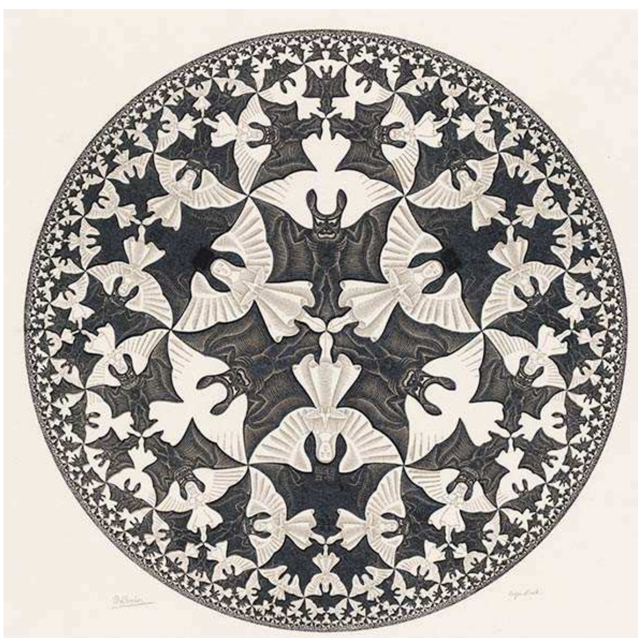


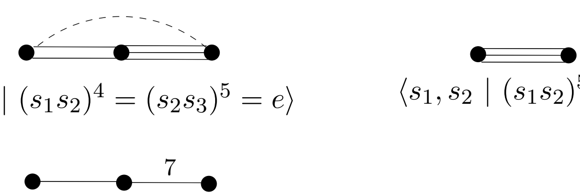
Figure 1. M.C.Escher, "Circle limit IV". (Image borrowed from the official webpage www.mcescher.com).

To work with Coxeter groups, it is convenient to encode the generators and relations by a Coxeter diagram.

Definition 1.2 (Coxeter diagram)**.** Let $G = \langle s_{,} \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle$ be a Coxeter group. By a *Coxeter diagram* of G we mean a graph with

- vertices $1, \dots n$ correspond to the generators $s_1, \dots, s_n$;
- edges: an edge between vertices i and j is labelled by m_{ij}.
  One usually uses simplified notation:
  – if $m_{ij} = 2$ then the edge between the vertices i and j is omitted;
  – if $m_{ij} = 3$ then the vertices i and j are joined by a single unlabelled edge;
  – if $m_{ij} = 4$ or 5 then the vertices i and j are joined by a double (resp. triple) edge without any label;
  – if $m_{ij} = \infty$ then one uses a dashed edge between i and j.

See Fig. 2 for examples.



$\langle s_1, s_2, s_3 \mid (s_1 s_2)^4 = (s_2 s_3)^5 = e \rangle$ $\langle s_1, s_2 \mid (s_1 s_2)^5 = e \rangle$

$\langle s_1, s_2, s_3 \mid (s_1 s_2)^3 = (s_2 s_3)^7 = (s_1 s_3)^2 = e \rangle$

FIGURE 2. Coxeter diagrams for some Coxeter groups.

Example 1.3 (Symmetry group of a regular pentagon)**.**

- *Symmetry group.* Consider a regular pentagon P_5 on $\mathbb{E}^2$ and denote by $Sym(P_5)$ the group of its symmetries. Clearly, $Sym(P_5)$ consists of 10 elements: every half-side of P_5 can be taken by a symmetry of P_5 to any other half-side, and this can be done in a unique way.
- *Generators of $Sym(P_5)$.* Let l_1 and l_2 be two lines through the centre of P_5 passing through some vertices of P_5 and forming an angle $\pi/5$ as in Fig. 3(a). Let s_1 and s_2 be reflections with respect to l_1 and l_2. Clearly, $s_1, s_2 \in Sym(P_5)$. Then the composition $s_1 s_2$ is a rotation by $2\pi/5$ around the centre of P. Therefore, s_1 and s_2 generate $Sym(P_5)$ (as applying rotation $s_1 s_2$ several times and composing with s_1 if needed one can take any half-side of P_5 to any other half-side).
- *Relations in $Sym(P_5)$.* We already know that $s_1^2 = s_2^2 = e$ (as s_i is a reflection) and $(s_1 s_2)^5 = e$ (as $s_1 s_2$ is a rotation by $2\pi/5$). In fact, any other relation in $Sym(P_5)$ follows from the relations above:
  - any word in s_1 and s_2 after applying relations $s_1^2 = s_2^2 = e$ (i.e. removing letters repeating twice) turns into an alternating word $(s_1)s_2 s_1 s_2 ....$;
  - moreover, such an alternating element will only preserve the orientation of $\mathbb{E}^2$ if the word contains even number of letters (i.e. when it is a power of the rotation $s_1 s_2$);
  - $(s_1 s_2)^k$ is the identity if and only if it is a power of $(s_1 s_2)^5$.

Hence, $Sym(P_5) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^5 = e \rangle$, which implies that $Sym(P_5)$ is a Coxeter group (with Coxeter diagram shown in Fig. 2 on the right).

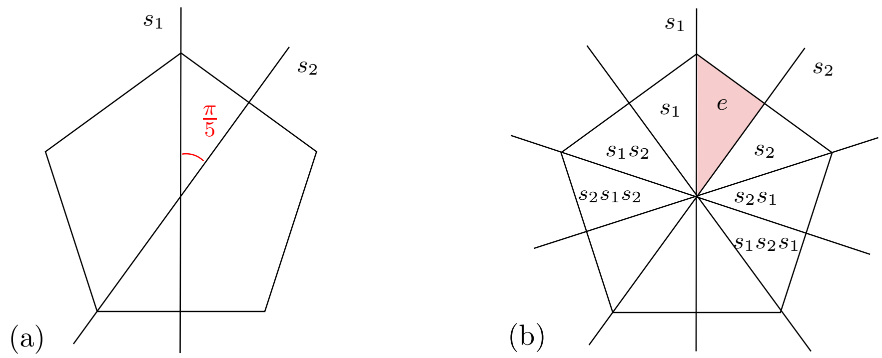


FIGURE 3. Symmetry group of regular pentagon: $\langle s_1, s_2 \mid s_i^2 = (s_1 s_2)^5 = e \rangle$.

Definition 1.4 (Reflection, reflection group)**.** Let X be one of the spaces $\mathbb{S}^d, \mathbb{E}^d$ and $\mathbb{H}^d$ (where $\mathbb{S}^d, \mathbb{E}^d$, $\mathbb{H}^d$ stay for spherical, Euclidean and hyperbolic d-dimensional spaces).

- By a *reflection* in X we mean an isometry of X preserving a hyperplane pointwise and swapping the half-spaces defined by the hyperplane. The hyperplane fixed by a reflection is called the *mirror* of the reflection.
- By a *reflection group* in X we mean a group generated by reflections in X. In these notes we will assume that there are *finitely many* generating reflections (though there are also infinitely generated reflection groups).

Definition 1.5 (Discrete group in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$)**.** Denote by $Isom(X)$ the group of isometries of X. A subgroup $G \subset Isom(X)$ is *discrete* if none of the orbits of G has accumulation points in X (i.e. if for every $x, y \in X$ there exists a ball N_x centred in x and containing finitely many points $\{g(y) \mid g \in G\}$).

Exercise 1.6. Let G be a reflection group in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ generated by reflections $s_1, \dots, s_n$. Let Π_i be the hyperplane fixed by s_i. Show that the angle $\angle a_{ij}$ between Π_i and Π_j is π-rational (i.e. $\alpha_{ij} \in \pi\mathbb{Q}$).

Hint: first, assume that $d = 2$, then derive the general statement from this case.

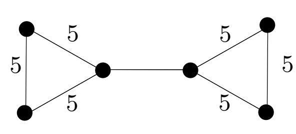
How does an action of a discrete reflection group G look like?

- The mirrors of reflections decompose X into congruent regions called *fundamental chambers*.
- Each fundamental chamber is an acute-angled polytope in X (i.e. a polytope with dihedral angles not exceeding $pi/2$).
- *Generators of G:* Choose a fundamental chamber F. Then G can be generated by reflections s_i with respect to the facets of F (where by a *facet* of F we mean a face of dimension $n-1$).
- *Relations in G:* The generating relations in G (other than $s_i^2 = e$) are in correspondence with dihedral angles of F, i.e.
  - given two facets Π_i and Π_j of F such that $\Pi_i \cap \Pi_j$ is an $(n-2)$-dimensional face of F, the corresponding reflections s_i and s_j satisfy $(s_i s_j)^{m_{ij}} = e$ for some $m_{ij} \in \mathbb{N}_{\geq 2}$ (compare to Exercise 1.6), and
  - any other relation in G follows from relations $s_i^2 = e$ and relations coming from dihedral angles of F.
- A dihedral angle between Π_i and Π_j is equal to π/m_{ij} if $m_{ij} \neq \infty$, otherwise $\Pi_i \cap \Pi_j = \emptyset$.
- The fundamental chambers in the tiling of X are in bijection with elements of G (see Fig. 3).
- Each fundamental chamber is a polytope with dihedral angles of size π/m_{ij} (such a polytope is called a *Coxeter polytope*).

Proposition 1.7. *Any discrete reflection group in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ is a Coxeter group.*

Definition 1.8. A Coxeter group G *acts by reflections* on $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ if every reflection $r \in G$ acts as a reflection of X.

Remark 1.9. (a) The statement converse to Proposition 1.7 is false: for example, one can show that the Coxeter group with Coxeter diagram



does not act by reflections in any of $\mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ (see Proposition 1.13).

(b) On the other hand, for every Coxeter group G there exists a piecewise Euclidean space (called *Davis complex*), where G acts as a discrete reflection group (see Construction 1.14).

Tits' representation (standard geometric representation)

Every Coxeter system (G, S) can be realised as a linear group in the following way. Let $G = \langle s_, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle$ be a Coxeter system. Let V be a vector space over $\mathbb{R}$ with basis $v_1, \dots, v_n$. Define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ by

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ -\cos \frac{\pi}{m_{ij}} & \text{if } m_{ij} < \infty \\ -1 & \text{if } m_{ij} = \infty \end{cases}$$

Consider the linear map $\sigma_i : V \to V$ defined by

$$\sigma_i(u) = u - 2\langle u, v_i \rangle v_i.$$

It is easy to check that σ_i preserves $\langle \cdot, \cdot \rangle$, $\sigma_i^2 = e$, $\sigma_i(v_i) = -v_i$, σ_i fixes the hyperplane $\{v \in V \mid \langle v_i, v \rangle = 0\}$ pointwise (i.e. σ_i can be considered as a linear analogue of a reflection).

Theorem 1.10 (Tits [T])**.** *The linear maps σ_i define a faithful representation*

$$\rho : G \to GL(n, \mathbb{R})$$

such that

- $\rho(s_i) = \sigma_i$
- *if $s_i \neq s_j$ then $\sigma_i \sigma_j$ has order m_{ij}.*

One can show that $\langle \cdot, \cdot \rangle$ is positive-definite if and only if G is finite. In this case, G acts by reflections in $\mathbb{S}^n$, and the action is generated by reflections with respect to facets of a simplex with dihedral angles π/m_{ij}.

To see that not every Coxeter group acts on one of the spaces $\mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ we will need the following definitions.

Definition 1.11 (Cosine matrix)**.** Let $G = \langle s_, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle$ be a Coxeter system. The $n \times n$ matrix $C = \{c_{ij}\}$ where

$$c_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\cos\frac{\pi}{m_{ij}} & \text{if } m_{ij} \neq \infty \\ -1 & \text{if } m_{ij} = \infty \end{cases}$$

is called a *cosine matrix* of G.

Definition 1.12 (Gram matrix)**.** Let $G = \langle s_, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle$ be a reflection group acting in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$. By a *Gram matrix* of G we will mean an $n \times n$ matrix $\{g_{ij}\}$ with

- $g_{ii} = 1$;
- $g_{ij} = g_{ji} = -\cos\frac{\pi}{m_{ij}}$ for all i, j such that $m_{ij} \neq \infty$;
- and g_{ij} defined for i, j such that $m_{ij} = \infty$ as follows:
  Assume that F is a fundamental chamber of the action of G shown in a linear model of X (i.e. in the vector space $\mathbb{E}^{d+1}$, $\mathbb{E}^d$ or $\mathbb{R}^{d,1}$ for $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ respectively, here $\mathbb{R}^{d,1}$ is a $(d+1)$-dimensional vector space with a quadratic form of signature $(d,1)$). Denote the scalar product on X by $\langle \cdot, \cdot \rangle$. Let $v_1, \dots, v_n$ be the unit outward normal vectors to the facets of F. Define
  $$g_{ij} = \langle v_i, v_j \rangle,$$
  i.e. $\{g_{ij}\}$ is the Gram matrix of vectors $v_1, \dots, v_n$. Notice that for $m_{ij} \neq \infty$ the formula $g_{ij} = \langle v_i, v_j \rangle$ recovers the definition $g_{ij} = -\cos\frac{\pi}{m_{ij}}$ given above.

Notice that if G acts in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ and $m_{ij} \neq \infty$ for all i, j then the Gram matrix coincides with the Coxeter matrix.
The following statement follows immediately from the definition of the Gram matrix.

Proposition 1.13. *Let $G = \langle s_, \dots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle$ be a reflection group acting in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ and let $Gr(G)$ be its Gram matrix. Then the the negative intertia index of $Gr(G)$ does not exceed 1.*

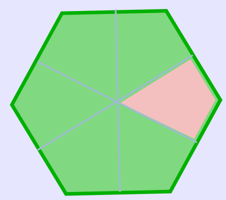
In particular, let G be the group discussed in Remark 1.9(a)), suppose it acts on $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$ by reflections. As $m_{ij} \neq \infty$, we see that the Gram matrix coincides with the Coxeter matrix $C(G)$, however, it is a straightforeward computation that the signature of $C(G)$ is $(4,2)$. This contradicts to Proposition 1.13 and, hence, shows that G does not act on X by reflecitons.

See [Vin1] for characterisation of reflection groups acting in $X = \mathbb{S}^d, \mathbb{E}^d$ or $\mathbb{H}^d$.

Construction 1.14 (Davis complex [D])**.** Davis complex for a Coxeter group G is a space where group G acts by reflections with compact fundamental domain. $\Sigma(G)$ is contractible, piecewise Euclidean complex (with $CAT(0)$ metric, i.e. it is a space looking similar to spaces of non-positive curvature).

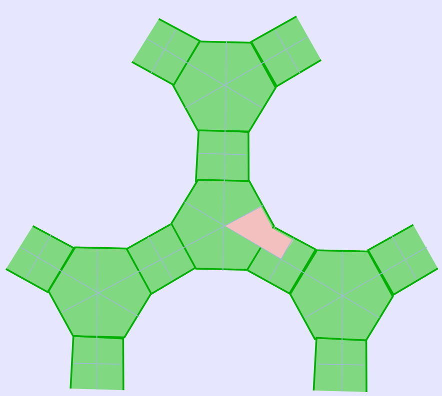
For a *finite* group G, the complex $\Sigma(G)$ is just one cell, which is obtained as a convex hull C of G-orbit of a suitable point p in the standard linear representation of G as a group generated by reflections. The point p is chosen in such a way that its stabilizer in G is trivial and all the edges of C are of length 1. The faces of C are identified with Davis complexes of the subgroups of G conjugate to subgroups generated by subsets of $s_1, \dots, s_n$.

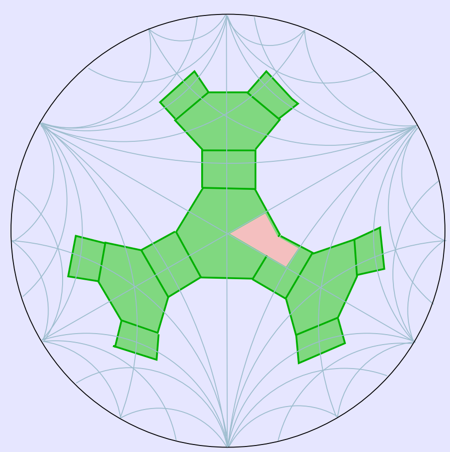
Example: Davis complex for $G = \langle s_1, s_2 \mid s_i^2 = (s_1 s_2)^3 = e \rangle$ is a regular hexagon:



If G is *infinite*, the complex $\Sigma(G)$ is built up of the Davis complexes of maximal finite subgroups of G glued together along their faces corresponding to common finite subgroups. Then the group G acts on $\Sigma(G)$ by reflections.

Example: Davis complex of the group $G = \langle s_1 s_2, s_3 \mid s_i^2 = (s_1 s_2)^2 = (s_2 s_3)^3 = e \rangle$ is glued of infinite number of regular Euclidean hexagons and quadrilaterals. The corresponding group also acts by reflections on hyperbolic plane $\mathbb{H}^2$ (so that the Davis complex can be quasi-isometrically embedded into $\mathbb{H}^2$).



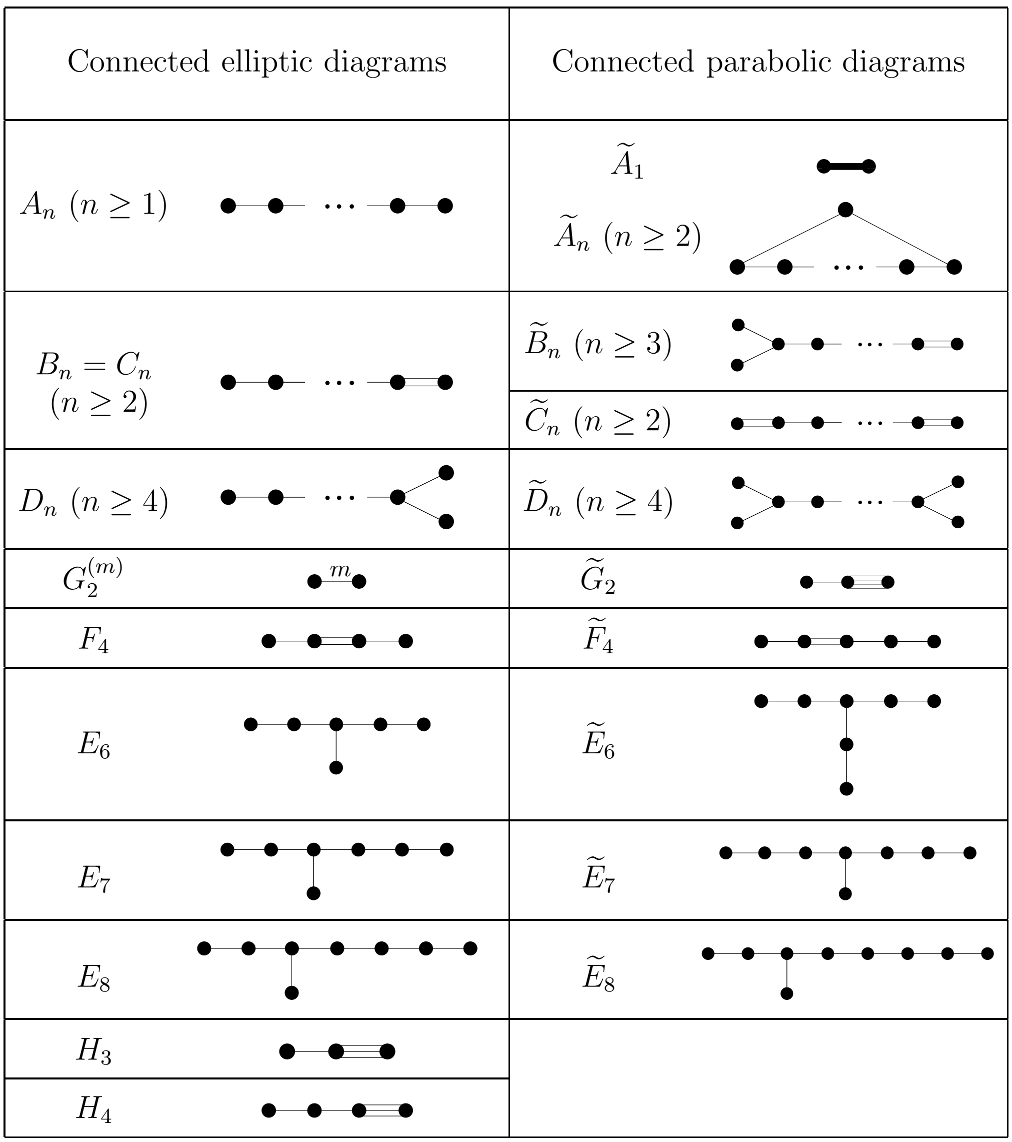


See [D] for the details.

1.2. **Discrete reflection groups in $\mathbb{S}^d, \mathbb{E}^d$ and $\mathbb{H}^d$.**

1. Discrete reflection groups in $\mathbb{S}^d$ (i.e. finite Coxeter groups). Due to compactness of $\mathbb{S}^d$ any group acting discretely (and effectively) on $\mathbb{S}^d$ should be finite. In particular, this is the case for Coxeter groups. On the other hand one can show that every finite Coxeter group acts on $\mathbb{S}^d$ for some d as a discrete reflection group. All finite Coxeter groups where classified by Coxeter [Cox]: their Coxeter diagrams are called *elliptic diagrams*, they are unions of diagrams shown in the left column of Table 1.1. A fundamental chamber for a discrete reflection group in S^n is always a simplex.

TABLE 1.1. Finite (left) and affine (right) Coxeter groups



Example 1.15. Consider a group $A_3 = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_3)^2\rangle$ with a Coxeter diagram •——•——• . Our aim is to see the action of A_3 on the sphere S^2. We will think of S^2 as embedded into $\mathbb{E}^3$:

- metric on S^2 is induced from $\mathbb{E}^3$,
- geodesics are great circles, i.e. intersections of S^2 with planes passing through the centre of the sphere,
- angles are measured as in $\mathbb{E}^3$.

The fundamental domain for the action of A_3 on S^2 will be a triangle with angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$. To see that such a triangle exists on S^2, consider a regular tetrahedron $ABCD$. Let O be its centre, H be the centre of the face ABC and M be the midpoint of BC, see Fig. 4. Consider the planes

$$\Pi_1 = (BMO), \qquad \Pi_2 = (BHO), \qquad \Pi_3 = (HMO),$$

where (XYZ) stays for a plane spanned by X, Y, Z. Suppose that O is also the centre of the sphere S^2. Then one can check from the symmetry that the intersection of S^2 with the planes Π_1, Π_2, Π_3 is a triangle with angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$ (where Π_1 is orthogonal to Π_3).

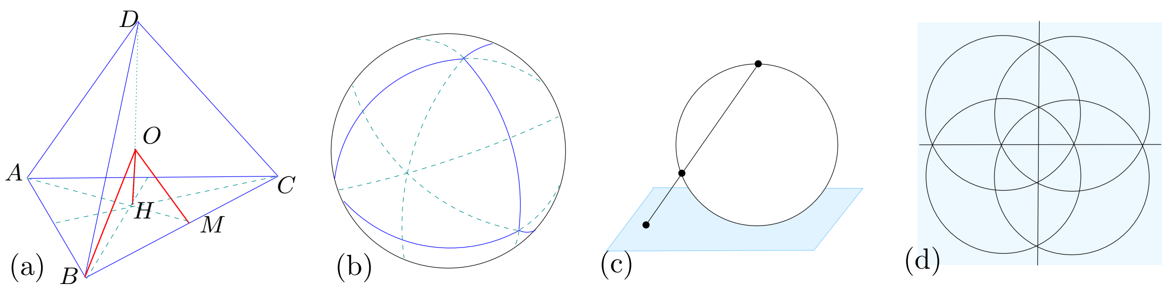


FIGURE 4. Action of A_3 on a sphere: (a) regular tetrahedron, (b) the same projected to a sphere, (c) stereographic projection, (d) tiling of the sphere by 24 triangles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$ seen in stereographic projection.

Notice, that tetrahedron $ABCD$ is tiled by 24 triangular cones (six based at each of the four faces). Projecting this tiling from the centre O to the sphere, we get a tiling of S^2 by 24 congruent spherical triangles. To be able to see this tiling better, we (stereographically) project the sphere to a plane (from a right vertex in one of the 24 triangles). The 24 triangles in the tiling correspond to the 24 elements of A_3 (which in its turn acts by permutations on the vertices of the tetrahedron $ABCD$). The group is generated by three reflections with respect to the sides of a fundamental domain.

Exercise 1.16. It is shown in Example 1.15 that the symmetry group of a regular tetrahedron coincides with the Coxeter group A_3. Find the symmetry groups (and their Coxeter diagrams) of other Platonic solids, i.e. cube, octahedron, dodecahedron and icosahedron.

2. Discrete reflection groups in $\mathbb{E}^d$ (i.e. affine Coxeter groups). Coxeter groups acting in Euclidean spaces are called *affine* Coxeter groups. These groups are also classified by Coxeter [Cox], their Coxeter diagrams are called *parabolic* and they are unions of the diagrams in the right column of Table 1.1. A fundamental chamber for an affine Coxeter group is either a simplex of a direct product of several simplices.

Example 1.17. Consider the group $\widetilde{A}_2 = \langle s_1, s_2, s_3 \mid s_i^2 = (s_i s_j)^3 = e \rangle$. It acts on the Euclidean plane and is generated by reflections with respect to the sides of a regular triangle (see Fig. 5).

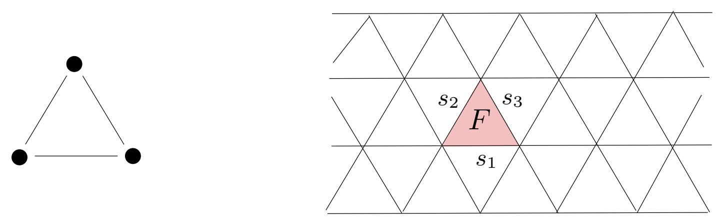


FIGURE 5. Coxeter diagram of $\widetilde{A}_2$ and the action of $\widetilde{A}_2$ on the Euclidean plane.

3. Discrete reflection groups in $\mathbb{H}^d$. There are infinitely many examples of such groups and there is no classification known. Fundamental chambers of discrete hyperbolic reflection groups are hyperbolic Coxeter polytopes (i.e. polytopes with dihedral angles of size π/m_{ij}).

Example 1.18. Consider a regular right-angled pentagon in $\mathbb{H}^2$ (i.e. a pentagon with all right angles). Gluing two copies of them along an edge one can obtain a right angled hexagon. Then gluing two copies of the hexagon we obtain a bigger right-angled polygon. Repeating this doubling trick many times we obtain right-angled polygons of various sizes and shapes, each of the polygons can serve as a fundamental chamber for a new reflection group.

In general, a Coxeter polytope in $\mathbb{H}^n$ may have

- very complicated combinatorial structure;
- very large number of facets;
- very small angles π/m_{ij} (with any large integer m_{ij}).

In particular, using the doubling trick one can show the following statement.

Theorem 1.19 ([All]). *There are infinitely many compact Coxeter polytopes in* $\mathbb{H}^d$ *for all* $d \leq 6$. *There are infinitely many finite volume Coxeter polytopes in* $\mathbb{H}^d$ *for all* $d \leq 19$.

At the same time, compact hyperbolic Coxeter polytopes do not exist in large dimensions:

Theorem 1.20 ([Vin2]). *There is no compact Coxeter polytopes in* $\mathbb{H}^d$ *in* $d > 29$.

Examples of compact Coxeter polytopes are known up to dimension $d \leq 8$ only. This leaves us with a number of open questions:

Open questions:

- Are there any compact hyperbolic Coxeter polytopes in dimensions 9-29?
- Are there only finitely many compact hyperbolic polytopes in dimensions 7 and 8? (To the moment we only know a unique example in dimension 8 and two examples in dimension 7).
- It is known that there are infinitely many examples up to dimension 6. However, most of them are obtained from several small examples by gluings (as in the doubling trick). Is there a finite number of polytopes such that all other compact hyperbolic polytopes can be obtained by gluings of copies of these ones?

The question of general classification of hyperbolic Coxeter polytopes turned out to be very difficult. There were a number of attempts to find partial classifications (for small dimensions, for small number of facets, for small values of m_{ij} in the sizes π/m_{ij} of dihedral angles, for small number of pairs of mutually non-intersecting facets, etc.).

Most of the works in this direction are based on the following observation [Vin1]. Given a Coxeter polytope P and its Coxeter diagram D_P (i.e. the Coxeter diagram of the group generated by reflections with respect to the facets of P), the faces of P correspond to *elliptic* subdiagrams of D_P. In other words, combinatorics of P is completely defined by knowing elliptic subdiagrams of D_P. Notice that the same information can be derived from the list of *minimal non-elliptic* subdiagrams of D_P (such diagrams are actually Coxeter diagrams of hyperbolic simplices, there are finitely many of them and the list is known due to Lanner, see [Lan]).

For more information and references concerning hyperbolic Coxeter polytopes see http://www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html

1.3. **Root systems.** Let $V = \mathbb{E}^n$ be a vector space with a scalar product $\langle \cdot, \cdot \rangle$. For a vector $v \in V$, denote by $v^\perp$ the hyperplane orthogonal to v. Then the reflection with respect to the hyperplane $v^\perp$ is a linear map $r_v : V \to V$ which can be written as

$$r_v(u) = u - 2\frac{\langle u, v\rangle}{\langle v, v\rangle} v \quad \forall u \in V.$$

It is easy to see that $r_v(u) = u$ for all $u \in v^\perp$ (i.e. r_v preserves the plane $v^\perp$) and $r_v(v) = -v$ (i.e. r_v swaps the half-spaces defined by $v^\perp$).

Definition 1.21 (Root system)**.** A (finite, reduced) *root system* in V is a finite non-empty subset $\Delta \subset V$ such that

(1) Δ spans V, $0 \notin \Delta$;
(2) if $\alpha \in \Delta$ then $r_\alpha \Delta \subset \Delta$;
(3) if $\alpha, \beta \in \Delta$, then $2\frac{\langle \alpha, \beta\rangle}{\langle \beta, \beta\rangle} \in \mathbb{Z}$;
(4) $\forall \alpha \in \Delta \ \ \forall \lambda \in \mathbb{R} : \quad \lambda\alpha \in \Delta \Leftrightarrow \lambda = \pm 1$.

The group W generated by all reflections r_α, $\alpha \in \Delta$ is a finite reflection group in $\mathbb{R}^n$, and so it is a Coxeter group. This reflection group is called *Weyl group* of Δ and denoted $W = W(\Delta)$, and it is a subgroup of the group of symmetries of Δ.

This implies that a Weyl group of a root system is a finite Coxeter group (however, there are no root systems of types H_3, H_4 and $G_2^{(m)}$ for $m \neq 3,4,6$). By the *type* of a root system one means the type of the corresponding finite Coxeter group. So, there are root systems of types $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$. The root systems B_n and C_n have the same Weyl group but differ by the lengths of roots: in B_n there are $n-1$ long roots and one short root, while in C_n there are $n-1$ short roots and 1 long root. Connected elliptic diagrams associated to root systems (together with an arrow indicating long and short roots in cases of B_n, C_n, G_2 and F_4) are called *Dynkin diagrams.*

Example 1.22. In dimension 1, there is a unique root system $A_1 = \{\alpha, -\alpha\}$ (up to multiplication by $\lambda \in \mathbb{R}$). All root systems in dimension 2 are $A_1 \times A_1$, $A_2, B_2 = C_2$ and G_2, they are shown in Fig. 6.

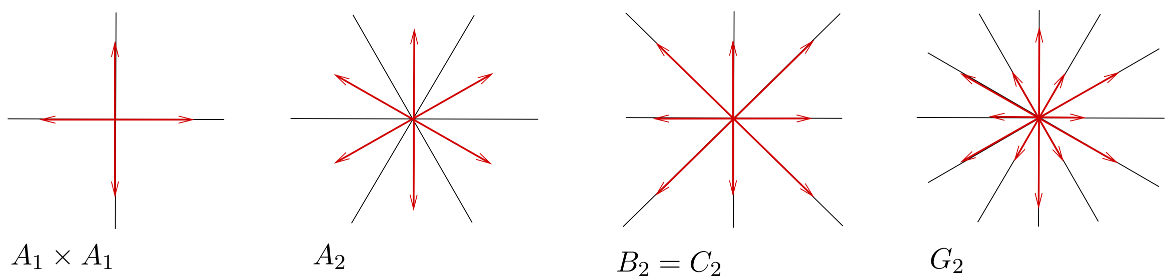


FIGURE 6. Finite root systems in dimension 2.

Definition 1.23 (Simple roots)**.** Hyperplanes $\{\alpha_i^\perp \mid \alpha_i \in \Delta\}$ decompose V into finitely many simplicial cones. Each of these cones is a fundamental chamber for the action of the Weyl group W. Choose one of the fundamental chambers, say C_0, and let $\alpha_1, \dots, \alpha_n \in \Delta$ be outward normals to the facets of C_0. Then $\alpha_1, \dots, \alpha_n$ are called *simple roots* of Δ.

Reflections with respect to the simple roots generate the Weyl group. So, all other reflections in Δ can be obtained as composition of reflections with respect to the simple roots. Condition (3) stating that $2\frac{\langle \alpha,\beta\rangle}{\langle \beta,\beta\rangle} \in \mathbb{Z}$ implies that

$$\forall \alpha \in \Delta, \quad \alpha = \sum_{i=1}^{n} k_i \alpha_i, \quad \text{where } k_i \in \mathbb{Z}.$$

Proposition 1.24. *For any $\alpha \in \Delta$, all k_i in the presentation $\alpha = \sum k_i \alpha_i$ are of the same sign.*

In other words, Δ consists of *positive* and *negative* roots:

Definition 1.25 (Positive roots)**.** A root $\alpha \in \Delta$ is *positive* if $\alpha = \sum k_i \alpha_i$ with positive k_i. Otherwise, α is *negative.*

Example 1.26.
- In the root system A_2, there are two simple roots $\Pi = \{\alpha_1, \alpha_2\}$ and three positive roots $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The whole system consists of 6 roots $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^- = -\Delta^+$. See Fig. 7(a).

- The root system A_3 contains three simple roots $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ and 6 positive roots $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. In Fig. 7 we label the mirrors of reflections of the Weyl group $W = W(A_3)$ by the corresponding positive roots.

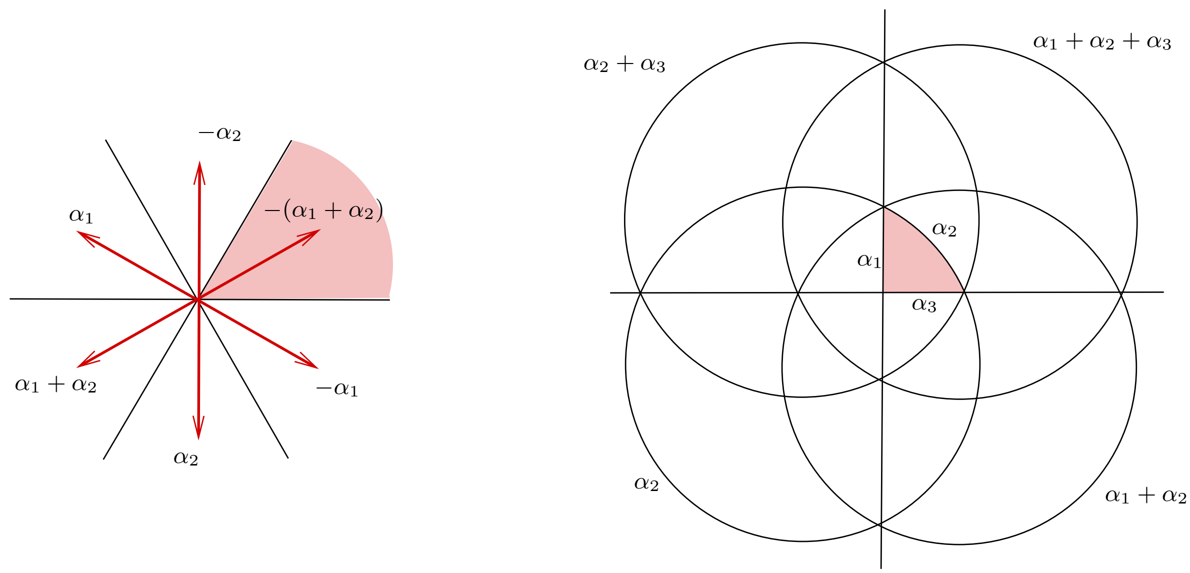


FIGURE 7. Root systems of type A_2 and A_3.

2. Lecture 2: Cluster algebras

Cluster algebras were introduced by Sergey Fomin and Andrei Zelevinsky in 2002 [FZ1]. A cluster algebra is defined by a combinatorial object like a quiver (i.e. an oriented graph) and is built from finite initial data by an iterative procedure called *seed mutation.* One can find an introduction to cluster algebras in [M], [W] or [FWZ1] and [FWZ2]. For more information on cluster algebras see *Cluster algebras portal* [F] by Sergey Fomin.

2.1. Quiver mutation.

Definition 2.1 (Quiver)**.** A *quiver* is an oriented weighted graph with integer weights on the edges. In the context of cluster algebras, one also assumes that a quiver contains

- no loops;
- no 2-cycles.

(This sort of quivers is sometimes called *cluster quivers*).

Remark 2.2. (a) We will always assume that a quiver has finitely many vertices and finitely many arrows, we denote the number of vertices by n and write $|Q| = n$.
(b) We will often label vertices by numbers $1, \dots, n$.
(c) p arrows from vertex i to vertex j is understood as $-p$ arrows from j to i.

Given a quiver Q, we can choose a vertex k and apply a local operation called *mutation of Q in direction k*: it will slightly change the quiver in a small neighbourhood of the vertex k and preserve it everywhere else.

Definition 2.3 (Quiver mutation)**.** Let Q be a quiver and $k \in Q$ be a vertex. By a *mutation* $\mu_k(Q)$ of Q in direction k one means the following procedure:

- reverse all arrows incident to k;
- for each oriented path $i \xrightarrow{p} k \xrightarrow{q} j$, with $p, q > 0$ in Q, if r is the number of arrows from j to i in Q, then $\mu_k(Q)$ will contain r' arrows from i to j, where

$$r + r' = pq,$$

as shown in Fig. 8:

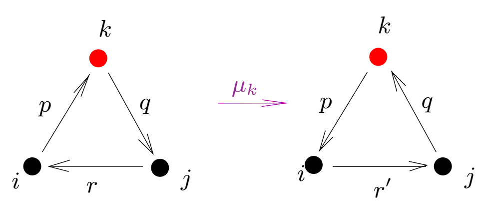


Figure 8. Quiver mutation μ_k: $r + r' = pq$ (here $p, q > 0$).

See Fig. 9 for an example of quiver mutation.

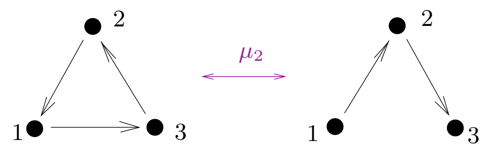


FIGURE 9. Example of quiver mutation.

Another way to represent a quiver is to write a skew-symmetric $n \times n$ matrix $B = \{b_{ij}\}$ with b_{ij} equal to the number of arrows from i to j.
Then the mutation μ_k is given by the following formula:

$$\mu_k(B) = B', \qquad \text{where} \qquad b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

The next proposition follows immediately from Definition 2.3:

Proposition 2.4. $\mu_k(\mu_k(Q)) = Q$.

Definition 2.5 (Mutation class of a quiver)**.**

- By a *mutation class* of a quiver Q we mean the set of all quivers which can be obtained from Q by application of finite sequences of mutations, i.e. $\{\mu_{i_s} \dots \mu_{i_2}\mu_{i_1}(Q) \mid i_k \in \{1, \dots, n\}, s \in \mathbb{N}\}$.
- Given a quiver Q' in the mutation class of Q, we will also say that Q' is *mutation equivalent to* Q and write $Q' \sim_\mu Q$.

One can visualise a mutation class of Q by drawing an n-regular tree (see Fig. 10) with quivers at the vertices and mutation as edges:

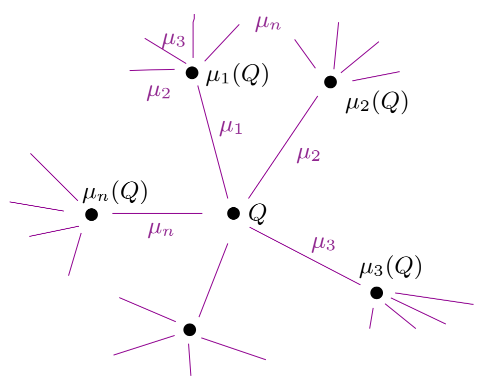


FIGURE 10. n-regular tree.

Some of the vertices of this tree may coincide. In most cases the mutation class is infinite, but sometimes it can be finite.

Definition 2.6 (Quiver of finite mutation type)**.** We say that a quiver Q is *of finite mutation type* or is *mutation-finite* if the mutation class of Q consists of finitely many quivers.

Question 2.7. When Q is of finite mutation type?

First quick answer: not too often, see Lemma 2.8.

Lemma 2.8. *If Q is a connected quiver, $|Q| > 2$ and Q contains $p_0 > 2$ arrows between some vertices i and j, then Q is mutation-infinite.*

Idea of Proof. Consider any connected subquiver $Q_3 \subset Q$ with 3 vertices containing the p-tuple arrow. Up to several mutations, we can assume that Q_3 is the following quiver $1 \xrightarrow{p} 2 \xrightarrow{q} 3 \xrightarrow{r} 1$ with $r \le q \le p$, $p \ge 2$, $q > 0$. Then

$$\mu_2(Q_3) = 1 \xleftarrow{p} 2 \xleftarrow{q} 3 \xleftarrow{r'} 1$$

where $r' = pq - r > 2q - r \ge r$. So, we increased the number of arrows in Q_3 (and preserved the p-tuple arrow). Repeating this we can obtain quivers with as many arrows as we want.

□

2.2. **Cluster algebra $\mathcal{A}(Q)$.** Let $x_1, \dots, x_n$ be indeterminants. Given a quiver with n vertices, we will associate to its vertices some rational functions $(u_1, \dots, u_n)$ in $x_1, \dots, x_n$.

Definition 2.9 (Seeds, seed mutation, cluster variables)**.**

- We start with the *initial seed* S_0 which is the pair $(Q, (x_1, \dots, x_n))$. (Here we think of x_i as being attached the vertex i of Q).
  Other seeds will be obtained from S_0 by applying mutations as follows.
- Given a seed $S = (\widetilde{Q}, (u_1, \dots, u_n))$, define the mutation of S in direction k by $\mu_k(s) = (\widetilde{Q}', (u_1', \dots, u_n'))$, where $\widetilde{Q}' = \mu_k(\widetilde{Q})$, $u_i' = u_i$ for all $i \ne k$ and
$$u_k' = \frac{1}{u_k}(\prod_{i \to k} u_i + \prod_{k \to j} u_j).$$
- All collections obtained from S_0 by iterated process of mutation will be called *seeds*, the functions $u_1, \dots, u_n$ in each seed will be called *cluster variables*.

Definition 2.10 (Cluster algebra $\mathcal{A}(Q)$)**.** The (coefficient free) cluster algebra $\mathcal{A}(Q)$ associated to the initial quiver Q is the algebra (a subalgebra of the field of rational functions in $x_1, \dots, x_n$) generated by the set of all cluster variables.

Remark 2.11. In general, a cluster algebra is generated by infinitely many cluster variables: we collect all cluster variables from all seeds.

Definition 2.12 (Exchange graph)**.** By an *exchange graph* of a cluster algebra $\mathcal{A}(Q)$ we mean a graph with vertices in bijection to seeds of $\mathcal{A}(Q)$ and edges between two vertices whenever there is a mutation between the corresponding seeds.

Example 2.13. Seed mutations in cluster algebra of type A_2 produce 5 seeds, containing altogether 5 cluster variables $x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}$ (see Fig. 11). Notice, that the two seeds drawn on the left both have x_1 at the source and x_2 at the sink of the arrow. So, these two seeds are identified.

Hence, the exchange graph of $\mathcal{A}(A_2)$ is a pentagon.

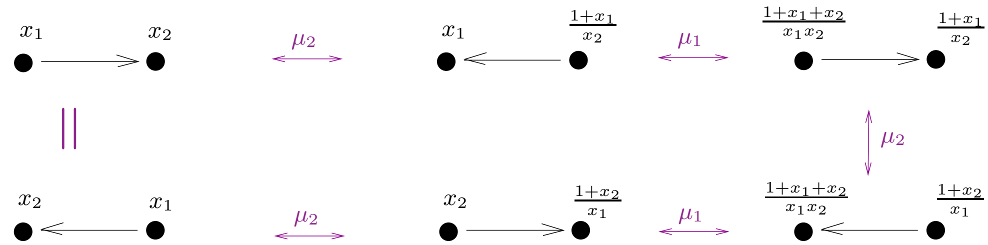


FIGURE 11. Type A_2: 5 seeds, 5 cluster variables. (The seed after five mutations coincide with the initial seed up to permutation of the vertices).

Example 2.14. One can check that the exchange graph of a cluster algebra of type A_3 is the graph shown in Fig. 12. This is also a 1-skeleton of 3-dimensional polyhedron called *associahedron*, see [FR] for more details and examples.

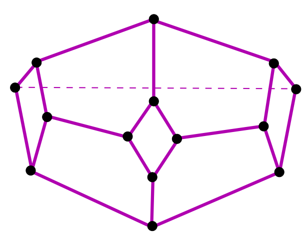


FIGURE 12. Exchange graph of $\mathcal{A}(A_3)$: 14 seeds.

Definition 2.15 (Cluster algebra of finite type)**.**

- A cluster algebra $\mathcal{A}(Q)$ is *of finite type* if the number of cluster variables in $\mathcal{A}(Q)$ is finite.
- A quiver Q is *of finite type* if $\mathcal{A}(Q)$ is of finite type.

Theorem 2.16 ([FZ2])**.**

- *A cluster algebra $\mathcal{A}(Q)$ is of finite type if and only if Q is mutation-equivalent to an orientation of a Dynkin diagram $\Delta = A_n, D_n, E_6, E_7$ or E_8.*
- *For a cluster algebra of type Δ,*
  - *initial cluster variables are in bijection with negative simple roots of Δ;*
  - *non-initial cluster variables are in bijection with positive roots of Δ.*

Positive roots together with negative simple roots are called *almost positive roots*. Using the notion of almost positive roots, one can state that cluster variables of a cluster algebra of finite type are in bijection with almost positive root of the corresponding root system.

Example 2.17 (Cluster algebra A_2)**.** Degrees of $x_1, \ldots, x_n$ in the denominators of cluster variables are encoded by almost positive roots of Δ:

initial variables:	x_1, x_2	$-\Pi = \{-\alpha_1, -\alpha_2\}$
non-initial variables:	$\frac{1+x_2}{x_1}, \frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1x_2}$	$\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$.

The following two properties of cluster algebras are formulated in [FZ1] and proved in [FZ1] and [LS] respectively:

- **"Laurent phenomenon"**: cluster variables are Laurent polynomials in $x_1, \dots, x_n$ (i.e. denominators are monomials);
- **"positivity":** numerators of cluster variables are sums of monomials with positive coefficients.

Proposition 2.18. *If $\mathcal{A}(Q)$ is of finite type then Q has finite mutation class.*

Proof. If the mutation class of Q is infinite, then there is a quiver $Q' \sim_\mu Q$ containing an arrow of multiplicity more than 2. Mutating repeatedly in the sink of that arrow one obtains already infinitely many cluster variables (in view of Theorem 2.16). $\square$

2.3. **Quivers of finite mutation type.** For a general quiver Q, there are infinitely many variables in $\mathcal{A}(Q)$ and infinitely many quivers in the mutation class of Q. For a quiver of finite type there are finitely many cluster variables and finitely many quivers in the mutation class. In addition, there may be some quivers of finite mutation type containing infinitely many variables (see Fig. 13).

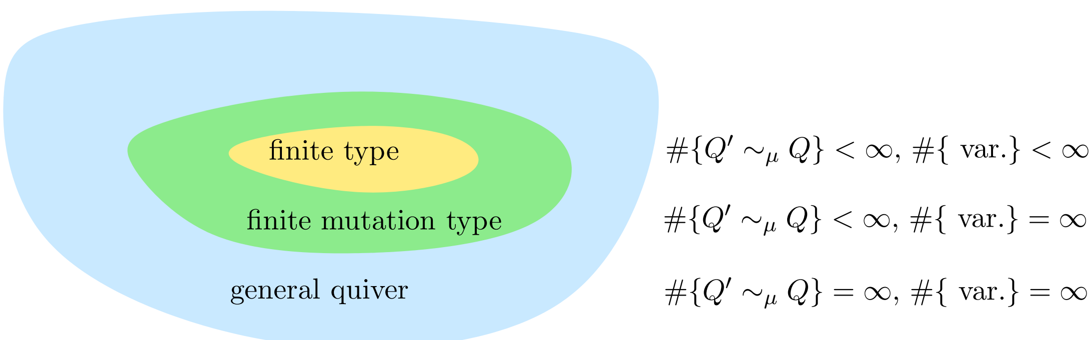
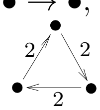


FIGURE 13. Types of cluster algebras.

Question 2.19. Are there quivers of finite mutation type which are not of finite type?

Example 2.20. Yes, there are:

1. $\bullet \xrightarrow{p} \bullet$, for $p \geq 2$.
2. 

3. Quivers arising from triangulated surfaces (see Section 2.4).

It was conjectured in [FST] that except for finitely many exclusions and rank 2 quivers, all quivers of finite mutation type are coming from triangulated surfaces.

2.4. **Quivers from triangulated surfaces.** The construction in this section comes from [FG] and [FST].

Let S be an orientable surface of some genus g, with (possibly empty) boundary ∂S and with marked points inside S and on ∂S. Assume that

- there is at least one marked point on each boundary component of S;
- the surface is triangulated in a way that the vertices of the triangles are in the internal and boundary marked points.

The internal edges of triangulations will be called *arcs*.

From a triangulation T on the surface S we construct a quiver Q as follows:

- vertices of Q correspond to internal arcs of the triangulation T,
- two vertices i and j are connected by an arrow $i \to j$ when the corresponding arcs lie in one triangle of the triangulation T and j-th arc follows i-th in the clockwise order (see Fig. 14).

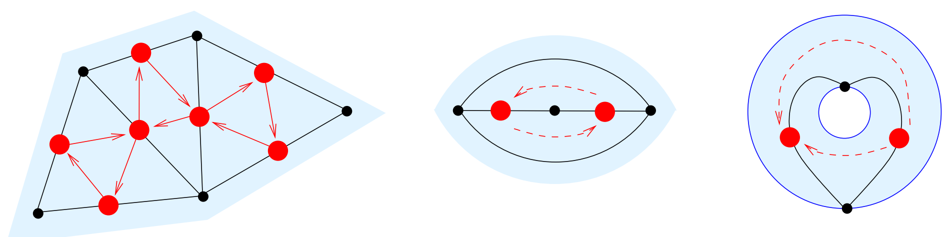
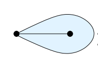


FIGURE 14. Left: quiver from a triangulated surface. Middle and Right: cancellation of arrows and a double arrow.

Notice that the same pair of arcs can lie in at most two different triangles, which can result in cancellation of arrows or in at most two arrows between two given vertices.

Remark 2.21. We allow self-folded triangles as the one here: ; in this case the arrows of Q incident to the vertex associated to the inner arc are constructed as follows: if i is the inner arc and l is the loop and k is some other arc, then the quiver Q will have an arc $i \xrightarrow{p} k$ if and only if Q has the arc $l \xrightarrow{p} k$.

Definition 2.22 (Flip)**.** Let γ be an arc of a triangulation T. Remove γ from T and replace it by the unique other arc γ' not crossing the arcs of $T \setminus \gamma$, see Fig. 15. This process is called a *flip* of γ.

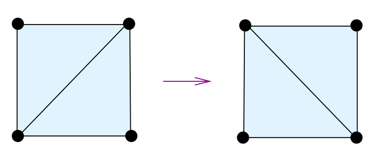


FIGURE 15. Flip of an arc.

The following statement is easy to check.

Proposition 2.23. *Let T be a triangulation, and T' be a triangulation obtained from T by the flip of the arc labelled k. Let Q and Q' be the quivers constructed from these triangulations. Then $\mu_k(Q) = Q'$.*

Theorem 2.24 ([H])**.** *Every two triangulations of the same surface are connected by a sequence of flips.*

This implies the following corollary.

Corollary 2.25.
- All quivers arising from triangulations of the same surface are mutation equivalent.
- Let S be a surface and T be a triangulation, let $Q = Q(T)$ be a quiver from T. Then quivers in the mutation class of T are in bijection with (combinatorial types of) triangulations of S.

Since the maximal multiplicity of an arrow in Q is 2, we get the following statement.

Proposition 2.26. *Quivers arising from triangulated surfaces are of finite mutation type.*

To be able to "flip" the internal arcs of self-folded triangles, one needs to introduce another version of triangulations called "tagged triangulations". We skip the details, see [FST].

2.5. Description of quivers from triangulations.

Example 2.27. Given two surfaces we can glue them along arcs isotopic to boundary. This will result in the gluing of the corresponding quivers, see Fig. 16

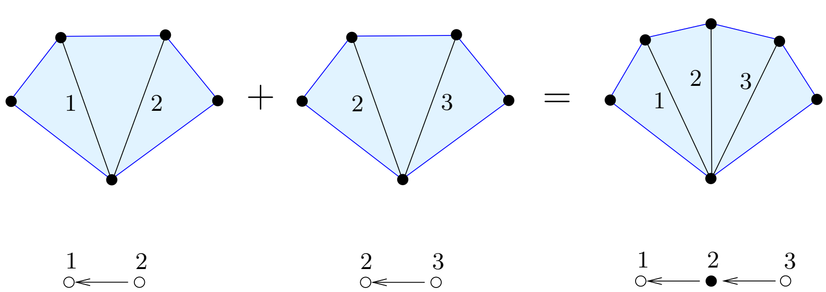


FIGURE 16. Gluing two surfaces. White nodes of quivers correspond to arcs where another surface may be attached.

Proposition 2.28 ([FST])**.** *Every triangulated surface can be glued out of the six surfaces shown in Fig. 17.*

The quivers corresponding to the surfaces in Fig. 17 are called *blocks*.

Definition 2.29 (Block-decomposable quiver)**.** A quiver Q is *block-decomposable* if it can be glued of finitely many blocks so that

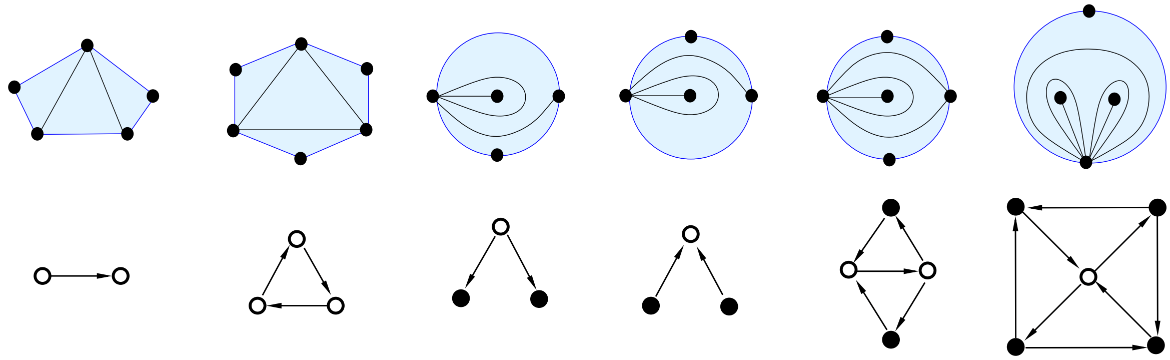


FIGURE 17. Blocks.

- a white vertex i of the block $\mathcal{B}_1$ may be identified with a white vertex of another block $\mathcal{B}_2$ if no other vertices (of any other blocks) are identified with i;
- suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are two blocks, and let $i_1 \xrightarrow{p_1} j_1$ be a subquiver of $\mathcal{B}_1$ and $i_2 \xrightarrow{p_2} j_2$ be a subquiver of $\mathcal{B}_2$; if the vertices i_1, j_1 are identified with the vertices i_2, j_2 respectively resulting in the vertices i and j of Q, then the number of arrows $i \to j$ in Q is equal to $p_1 + p_2$.

Otherwise, the quiver is called *non-decomposable*.

The definition together with Proposition 2.28 imply the following corollary.

Corollary 2.30. A quiver Q arises from a triangulated surface if and only if Q is block-decomposable.

Example 2.31 (Markov quiver)**.** The quiver shown on Fig. 18 can be obtained by attaching two triangular blocks along all three vertices (this exactly results in double arrows between each pair of vertices). The corresponding surface is a punctured torus. The quiver is called *Markov quiver* because of its tight connection to Markov equation $x^2 + y^2 + z^2 = 3xyz$ (see [BBH]).

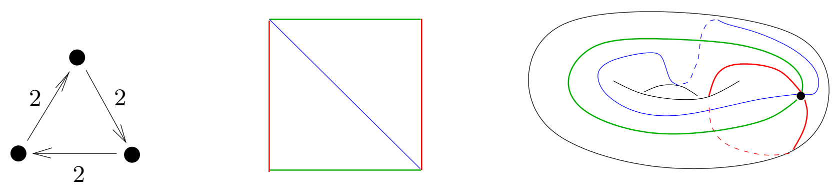


FIGURE 18. Markov quiver as a gluing of two triangular blocks.

2.6. **Classification of quivers of finite mutation type.** It was shown above that all quivers arising from triangulated surfaces are mutation-finite. Now, our aim is to classify all other mutation-finite quivers. In other words, we need to find all mutation-finite quivers which are not block-decomposable.

Remark 2.32. (Inspired by hyperbolic Coxeter polytopes)
Recall that while working with hyperbolic Coxeter polytopes we were based on the following observations:

- finite Coxeter groups are known;
- combinatorics of a Coxeter polytope can be described by looking at all elliptic subdiagrams of its Coxeter diagram (where by elliptic diagrams we mean diagrams of finite Coxeter groups);
- hence, combinatorics of a Coxeter polytope can be described by looking at all *minimal non-elliptic* subdiagrams of its Coxeter diagram.

Idea: to classify *minimal* (by inclusion) quivers which are *not block-decomposable.*

Remark 2.33. It makes sense to speak about minimal non-decomposable quivers, since a subquiver of a block-decomposable quiver is block-decomposable. Indeed, given a surface S, a triangulation T, an arc $\gamma \in T$ and a quiver $Q = Q(T)$, we can cut S along an arc γ and construct a quiver from the new surface $S \setminus \gamma$ (with triangulation $T \setminus \gamma$), we will obtain the quiver $Q \setminus v$, where v is the vertex of Q associated to γ.

Example 2.34. It is easy to check that the quivers in Fig. 19 are minimal non-decomposable (i.e. they are non-decomposable, while every proper subquiver of them is block-decomposable).

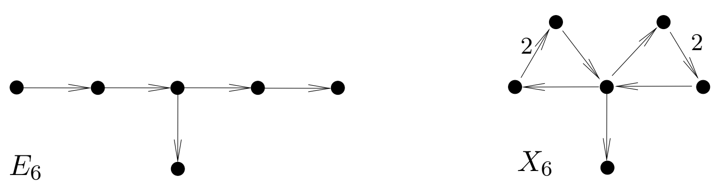


FIGURE 19. Quivers E_6 and X_6 are minimal non-decomposable.

Lemma 2.35 ([FeSTu])**.**

- *If Q is minimal non-decomposable, then $|Q| \leq 7$.*
- *If Q is minimal non-decomposable mutation-finite, then Q is mutation equivalent either to E_6 or to X_6.*

Theorem 2.36 ([FeSTu])**.** *A quiver Q is of finite mutation type if and only if one of the following holds:*

- *either $|Q| = 2$;*
- *or Q arises from a triangulated surface;*
- *or Q is mutation equivalent to one of the eleven quivers shown in Fig. 20.*

Open question: Theorem 2.36 gives a classification of quivers of finite mutation type, however the proof leaves unclear the following:

- is there any deep reason for the exceptional (non-surface) quivers to be in the list? (Is there any other way to characterise this list?)

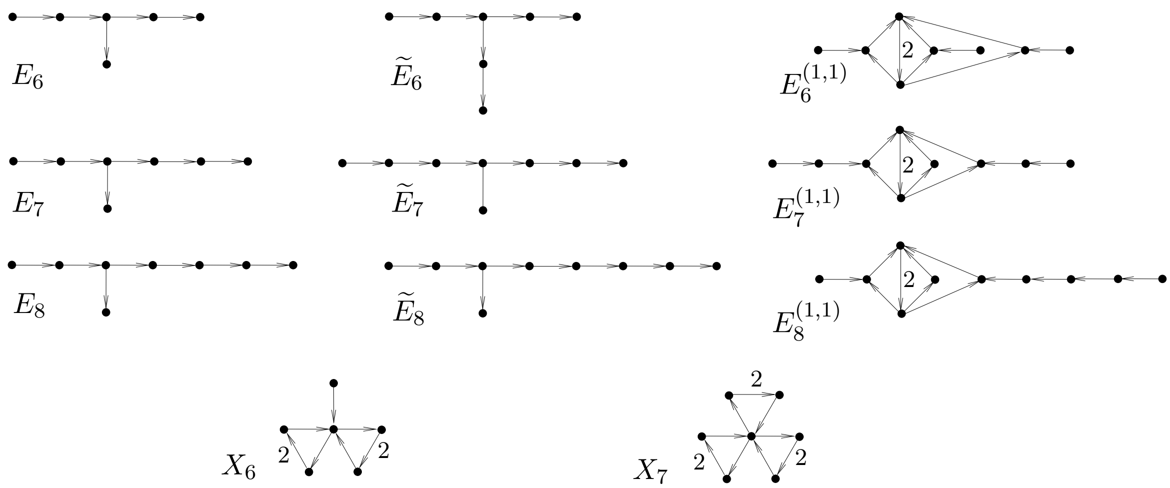
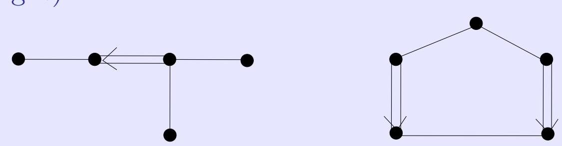


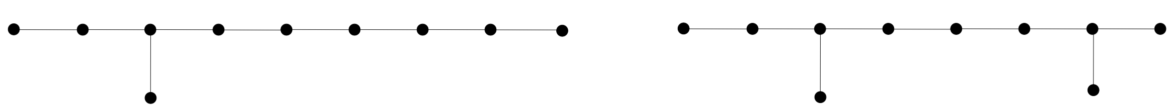
FIGURE 20. Exceptional mutation finite quivers.

Exceptional quivers E_6^*, E_7^*, E_8^* in Theorem 2.36 are clearly related to corresponding root systems.
It is less obvious that the quiver X_7 is also related to a root system: its underlying graph can be obtained as a Coxeter diagram for the subgroup generated by reflections in all short roots of the root system with Dynkin diagram on the left (or long root of the diagram on the right).



Proposition 2.37 ([FeSTu])**.** *Any minimal mutation-infinite quiver has at most 10 vertices.*

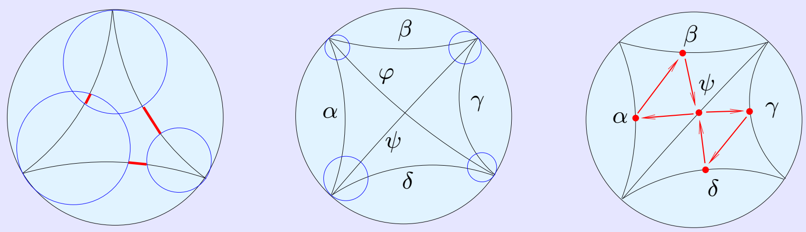
Example 2.38. There are minimal mutation-infinite quivers with 10 vertices given by orientations of the following Coxeter diagrams of finite volume hyperbolic simplices in $\mathbb{H}^9$:



Remark 2.39. Any Coxeter diagram of a hyperbolic (arithmetic) simplex can be oriented so that the constructed quiver is minimal mutation-infinite. The converse is not true: not every minimal mutation-infinite quiver is mutation equivalent to an orientation of a Coxeter diagram of a finite volume hyperbolic simplex (see [L]).

2.7. **Cluster variables for triangulated surfaces.** One can use hyperbolic metric on the surface to find a geometric interpretation for cluster variables [P]:

1. Assume that every triangle in the triangulation is an "ideal hyperbolic triangle", i.e. a triangle isometric to a triangle $t \subset \mathbb{H}^2$ with all three vertices on the boundary of $\mathbb{H}^2$.
2. The arcs of the triangulation then have infinite hyperbolic length (so, the length of the arc can not be immediately used for the geometric interpretation).
3. Cut the arcs by horocycles centred at the vertices of the triangle, see the left figure (here by a horocycle we mean a curve h represented in the Poincaré disc model of $\mathbb{H}^2$ by a circle tangent to the boundary $\partial\mathbb{H}^2$, the point $h \cap \partial\mathbb{H}^2$ is called the *centre* of the horocycle h).



4. Then, what is left of any given arc γ has finite hyperbolic length l_γ (if the horocycles at the ends of the arc intersect each other, we consider l_γ negative).
5. Denote $x_\gamma = exp(l_\gamma/2)$ (this value is called a *lambda length* of the arc γ).
6. The lambda lengths of arcs in a quadrilateral (see the middle figure) satisfy Ptolemy relation:
$$x_\varphi x_\psi = x_\alpha x_\gamma + x_\beta x_\delta.$$
7. This relation exactly coincides with the exchange relation for cluster variables associated to the arcs in the quadrilateral (see the right figure for the corresponding quiver).

Theorem 2.40 ([FT]). *Every cluster gives a coordinate on the decorated Teichmuller space of S.*

In other words, given the n-tuple $x_1, \dots, x_n$ of cluster variables lying in one cluster, one can reconstruct the hyperbolic metric on the triangulated surface S together with the choice of horocycles at the marked points).

Remark 2.41. Theorem 2.40 is based on the following easy to show fact:

> *Given positive numbers x_1, x_2, x_3, there exists a unique (up to isometry of $\mathbb{H}^2$) choice of an ideal hyperbolic triangle ABC and horocycles centred at A, B, C such that x_1, x_2, x_3 coincide with the lambda lengths of arcs AB, BC, CA (computed using the chosen horocycles).*

3. Lecture 3: Cluster algebras and Coxeter groups

3.1. **Coxeter group from a quiver of finite type.** Recall that by a quiver of finite type we mean a quiver mutation-equivalent to an orientation of a Dynkin diagram (see Theorem 2.16). In [BM] Barot and Marsh have constructed a group $W(Q)$ for every quiver Q of finite type in the following way:

- *Generators* $s_1, \dots, s_n$ of $W(Q)$ are in bijection with the vertices $v_1, \dots, v_n$ of Q;
- *Relations:* there are three types of relations:
  - (R1) $s_i^2 = e$ for $i = 1, \dots, n$;
  - (R2) $(s_i s_j)^{m_{ij}} = e$, for all $1 \le i < j \le n$, where $m_{ij} = 2$ if v_i and v_j are disjoint in Q and $m_{ij} = 3$ if v_i and v_j are joined by a single arrow (otherwise, $m_{ij} = \infty$, i.e. no relation produced);
  - (R3) cycle relation: for each oriented chordless cycle

$$v_{i_1} \to v_{i_2} \to \dots \to v_{i_{m-1}} \to v_{i_m} \to v_{i_1}$$

in Q add a relation

$$(s_{i_1} \quad s_{i_2} s_{i_3} \dots s_{i_n} \dots s_{i_3} s_{i_2})^2 = e.$$

Remark 3.1.
- Notice that relations (R1) and (R2) define a Coxeter group W_0 (its Coxeter diagram coincides with the quiver Q whose arrows are substituted by unoriented edges).
- The cycle relations (R3) look as $(r_i r_j)^k$, where r_i and r_j are reflections in W_0 (i.e. elements conjugate to the generating reflections s_i).

Theorem 3.2. [BM]

(a) *Given a quiver Q of finite type, the group $W(Q)$ is invariant under mutations of Q, i.e. $W(Q) \cong W(\mu_k(Q))$ for all $k = 1, \dots, n$.*

(b) *In particular, $W(Q)$ is a finite Coxeter group.*

(c) *If $Q' = \mu_k(Q)$ and s_i are generators of $W(Q)$ satisfying (R1)-(R3), then*

$$t_i = \begin{cases} s_k s_i s_k & \text{if } k \to i \text{ in } Q \\ s_i & \text{(otherwise)} \end{cases}$$

are generators of $W(Q')$ satisfying (R1)-(R3).

Idea of proof. Change the generators as in (c) and show that the relations (R1)-(R3) for s_i imply the relations (R1)-(R3) for t_i (and backwards). This will show parts (a) and (c). To prove part (b), notice that Dynkin diagrams contain no cycles, so if Q is an orientation of a Dynkin diagram then there are no cycle relation in the definition of $W(Q)$, which implies that $W(Q)$ is the corresponding Coxeter group.

□

3.2. **Geometric interpretation: A_3 case.** Consider the group $W(Q)$ discussed above for the case of $Q = A_3$, i.e. $1 \to 2 \to 3$. We get the Coxeter group

$$W \cong W(Q) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle.$$

This is a finite Coxeter group of type A_3 acting on a sphere by reflections as in Example 1.15.

Now, let $Q' = \mu_2(Q)$ (see Fig. 21(a)). Then

$$W = W(Q') = \langle t_1, t_2, t_3 \mid t_i^2 = (t_it_j)^3 = (t_1\ t_2t_3t_2)^2 = e\rangle.$$

Denote by W_0 the group $\langle t_1, t_2, t_3 \mid t_i^2 = (t_it_j)^3 = e\rangle$ (same generators and almost the same relations as in $W(Q')$ but no cycle relation). Notice that W_0 is an affine Coxeter group acting on $\mathbb{E}^2$ by reflections (and generated by reflections with respect to the sides of a regular triangle as in Example 1.17, see Fig. 21(b)). Note that the plane is tessellated by infinitely many copies of the fundamental domain, each copy F_g labelled by an element $g \in W_0$. To see the action of the initial group $A_3 = W(Q)$, we need to take a quotient by the cycle relation $(t_1\ t_2t_3t_2)^2 = e$. This means that in the tessellated plane we identify F_e with the fundamental triangle labelled by $(t_1\ t_2t_3t_2)^2$.

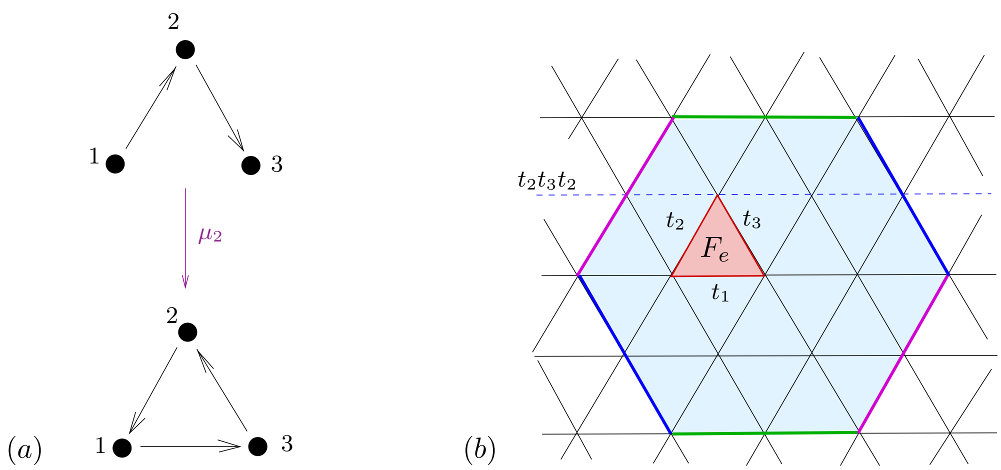


FIGURE 21. Mutation μ_2 and action of $W = W(Q') = A_3$ on the torus: 24 regular triangles tile the hexagon whose opposite sides are identified.

Now, let us see what does the transformation $T_1 = (t_1\ t_2t_3t_2)^2$ mean geometrically. The element t_1 acts on the plane as the reflection in the side l_1 of F_e, $t_2t_3t_2$ is the reflection with respect to the line l_{232} parallel to l_1 and passing through the vertex of F_e not contained in l_1. Denote by d the distance between l_1 and l_{232}. Then the transformation $(t_1\ t_2t_3t_2)$ is a translation of the plane by the distance $2d$ in the direction orthogonal to l_1. Hence, the element $T_1 = (t_1\ t_2t_3t_2)^2$ is the translation by $4d$ in the direction orthogonal to l_1.

Notice that $e = t_2(t_1\ t_2t_3t_2)^2t_2 = (t_2t_1t_2\ t_3)^2$ and $e = t_3(t_1\ t_2t_3t_2)^2t_3 = (t_3t_1t_3\ t_2)^2$ in W. This implies that we also need to take a quotient of the plane by the actions of the elements $T_2 = (t_3t_1t_3\ t_2)^2$ and $T_3 = (t_2t_1t_2\ t_3)^2$, i.e. by the translations by $4d$ in directions orthogonal to the lines l_2 and l_3. The quotient of the plane by the group W_{rel} generated by three translations T_1, T_2, T_3 is a hexagon with 3 pairs of opposite sides identified, i.e a flat torus.

Thus, we obtain an action of the quotient group $A_3 = W = W(Q')$ on the torus: the fundamental domain of this action is a regular triangle, 24 copies of this triangle tile the torus (as 24 copies tile the hexagon).

3.3. **Geometric interpretation: general settings.** The story shown in Section 3.2 for A_3 can be retold for every quiver Q of finite type:

- For a quiver Q, one can consider the group $W(Q)$ with relations (R1)-(R3) and the Coxeter group $W_0 = W_0(Q)$ with relations (R1) and (R2).
- For each Coxeter group G there exists a special space called *Davis complex* and denoted $\Sigma(G)$ such that G acts by reflections on $\Sigma(G)$ (see Construction 1.14).
- Consider $\Sigma(W_0)$ and take its quotient by the cycle relations (let $W_{rel} := NCl(\mathrm{R3})$ be the normal closure of the union of relations of type (R3) in W_0).
- Then $W = W(Q)$ acts on $X = \Sigma(W_0)/W_{rel}$.

Theorem 3.3 ([FeTu1], Manifold Property)**.** *The group W_{rel} is torsion-free. In particular, if $\Sigma(W_0)$ is a manifold then X is also a manifold.*

In other words, the space X is as good as $\Sigma(W_0)$ and by taking the quotient we are not introducing any new singularities.

This allows us to construct hyperbolic manifolds with large symmetry groups.

Example 3.4. Consider a mutation of the quiver Q of type A_4 as in Fig. 22(a). The quiver $Q' = \mu_3(Q)$ is an orientation of a Coxeter diagram of a non-compact hyperbolic simplex. One can show that the procedure above results in a 3-manifold with 5 cusps, and the constructed manifold admits an action of the group A_4.

Similarly, a quiver of type E_8 is mutation-equivalent to the quiver in Fig. 22(b), the undirected version of this quiver is a Coxeter diagram of 7-dimensional simplex. This results in a 7-dimensional manifold (with 2160 cusps) with action of the Coxeter group E_8. See [FeTu1] for more examples.

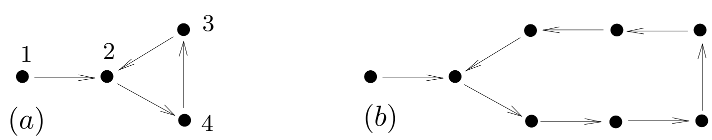


FIGURE 22. Quivers mutation-equivalent to A_4 (a) and E_8 (b).

Remark 3.5. In general, it is not an easy task to construct a hyperbolic manifold with a large symmetry group: for this one would need to take a group G acting on $\mathbb{H}^n$ discretely, freely, with finite volume fundamental domain, and having a torsion-free finite index subgroup $H \triangleleft G$, then $\mathbb{H}^n/H$ is a hyperbolic manifold with action of G/H. However, in general it is not easy to find such groups G and H.

3.4. **Beyond finite case.** One can prove an analogue of Theorem 2.16 in the following settings [FeTu2]:

- for affine quivers;
- for quivers from unpunctured surfaces;
- exceptional mutation-finite quivers;
- quivers from punctured surfaces of genus 0.

(In some of these cases one needs some additional relations, all of the form $(r_ir_j)^2 = e$, where $r_i = ws_iw^{-1}$ is a reflection in the corresponding Coxeter group W_0.)

What is different to finite type case:

- The group $W = W_0/W_{rel}$ is still a Coxeter group in the affine case, but it is some *quotient* of a Coxeter group otherwise;
- Manifold property still holds for affine case, but is not known otherwise.

In particular, in the settings of triangulated surfaces a theorem similar to Theorem 2.16 states that the group $W = W(Q)$ is an invariant of the surface. However, it is not known how good are these invariants:

Open questions: In the surface case, the group $W = W(Q)$ provides an invariant of the surface. How good is the invariant?

- are the groups different for different surfaces?
- what kind of groups are they? (when are these group finite?)
- are all these groups non-trivial at all?

3.5. **Back to the action of A_3 on S^2.** Theorem 2.16 describes different generators ($s_1, \dots, s_n$ and $t_1, \dots, t_n$) for the same group W. In the previous sections we used different spaces to illustrate actions of W for different choices of generators. However, we can look at different generators of W in the same space (say, S^n in the case of a finite group). We will consider the case of $W = A_3$:

Initial seed: Let s_1, s_2, s_3 be the generators coming from the quiver $1 \to 2 \to 3$ (denote this quiver Q). Then s_1, s_2, s_3 can be seen as reflections with respect to the sides of a spherical triangle T with angles $(\pi/2, \pi/3, \pi/3)$, see Fig. 23(a).

Mutation: Applying mutation μ_2 to Q we get the generators $t_1 = s_1, t_2 = s_2, t_3 = s_2s_3s_2$ (subject to relations $t_i^2 = (t_it_j)^3 = e$). As s_1, s_2, s_3 act by reflections with respect to the sides l_1, l_2, l_3 of the triangle T, the element $t_3 = s_2s_3s_2$ is a reflection with respect to some line l_{232} (where l_{232} is a reflection image of l_3 with respect to l_2). So, the generators t_1, t_2, t_3 correspond to a triangle bounded by l_1, l_2, l_{232}, see Fig. 23(b).

All seeds together: We can continue mutating the quiver Q moving through all seeds in the cluster algebra $\mathcal{A}(Q)$, we construct generators for W defined by each of the seeds and draw a triangle for each generating set. This results in the tiling of S^2 by 14 spherical triangles shown in Fig. 23(c). The tiling is dual to the associahedron (see Fig. 23(d) and compare to (e)).

Remark 3.6. For every spherical triangle showing up as a seed in Fig. 23(c) we can consider inward normal vectors to its sides (these are some roots of the root system A_3). In particular, for the initial seed these are negative simple roots. Under the mutation these vectors change as follows: if v_1, v_2, v_3 are the inward normals for the seed S with the quiver Q, then the inward normals v_1', v_2', v_3' for the seed $S' = \mu_k(S)$

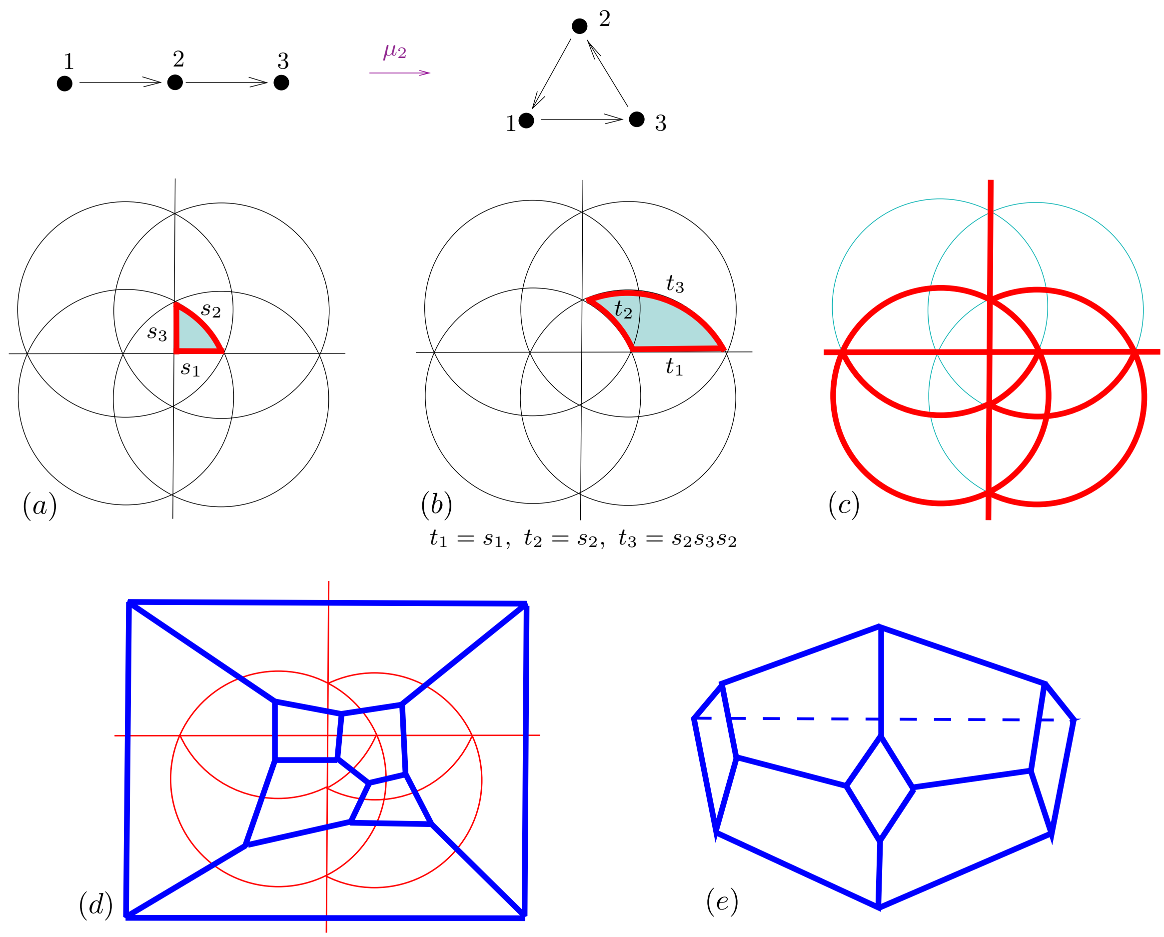


FIGURE 23. Different sets of generators of $W = A_3$ as clusters of $\mathcal{A}(A_3)$.

may be computed by

$$v_i' = \begin{cases} -v_i & \text{if } i = k \\ r_{v_k}(v_i) = v_i - \frac{\langle v_i, v_k \rangle}{\langle v_i, v_i \rangle} v_k & \text{if } k \to i \text{ in } Q \\ v_i & \text{otherwise} \end{cases}$$

where by $r_{v_k}(v_i)$ we mean a reflection of v_i with respect to the hyperplane $\langle u, v_k \rangle = 0$ orthogonal to v_k.

3.6. **Beyond finite mutation type: acyclic quivers.**

Definition 3.7 (Acyclic quiver)**.** A quiver Q is called *acyclic* if it contains no oriented cycles. A quiver Q is *mutation-acyclic* if there is an acyclic quiver in the mutation class of Q.

The situation described in Section 3.5 can be generalised to mutation classes of acyclic quivers in the following way (see also [BGZ] and [S]).

Initial configuration:

- *Matrix:* Given an acyclic quiver Q with b_{ij} arrows from i to j, consider a symmetric matrix $M = \{m_{ij}\}$ where $m_{ii} = 2$, $m_{ij} = -|b_{ij}|$. This matrix defines a quadratic form and we can consider M as a Gram matrix (i.e., the matrix of inner products) of some n-tuple of basis vectors $(v_1, \dots, v_n)$ in a quadratic n-space V of the same signature as M has.
- *Reflection:* Given a vector $v \in V$ with $\langle v, v\rangle = 2$, consider a *reflection*

$$r_v(u) = u - \langle u, v\rangle$$

  with respect to $\Pi_v = v^\perp$, where $\langle \cdot, \cdot\rangle$ is the inner product defined by the quadratic form. One can easily check that r_v preserves the quadratic form in V and that $r_v(v) = -v$, i.e. that r_v is an (pseudo-)orthogonal transformation preserving Π_v and interchanging the half-spaces into which V is decomposed by Π_v.
- *Reflection group:* We denote by W the group generated by reflections $s_1 = r_{v_1}, \dots, s_n = r_{v_n}$ in hyperplanes $\Pi_i = \Pi_{v_i}$. According to [Vin3], G acts discretely in some cone $C \subset V$ with fundamental chamber $F = \cap_{i=1}^n \Pi_i^-$, where $\Pi_i^- = \{u \in P(V) \mid \langle u, v_i\rangle < 0\}$. The fundamental chamber can also be understood as a connected component of the complement of the mirrors of all reflections inside C. (The images of vectors v_i under W are precisely real roots of the root system Δ constructed by the generalized Cartan matrix $M(B)$, the vectors v_i are simple roots, G is the corresponding Weyl group.)

Summarising, given an acyclic quiver Q we have constructed a reflection group $W = W(Q)$ with a specified set of generating reflections $s_1, \dots, s_n$. Notice that we can do this for any quiver, not just for acyclic ones (but the next step will work well for mutation acyclic quivers only).

Mutation: Let $Q' = \mu_k(Q)$ and Q is mutation acyclic quiver. Let $s_1, \dots, s_n$ be the generating reflections of $W(Q)$. Consider another generating set $t_1, \dots, t_n$ of reflections for $W = W(Q)$, where

$$t_i = \begin{cases} s_k s_i s_k & \text{if } k \to i \text{ in } Q \\ s_i & \text{otherwise} \end{cases}$$

Claim. Then $t_1, \dots, t_n$ is the generating set for $W(Q')$ (and hence $W(Q') = W(Q)$).

This privides a geometric realisation of quiver mutation for acyclic quivers.

Open question: Is it possible to find a geometric/combinatorial realisation for mutations of a general quiver?
(In ranks 2 and 3 the answer is positive [FeTu3], but already the case of rank 4 is not known).

Conclusion

We have seen that cluster algebras are connected to Coxeter groups in a number of ways including:

- in *finite types*, cluster variables are in bijection with almost positive roots in the root system of the corresponding type ([FZ2]);
- classification of *mutation finite* types is in implicit connection with classification of Coxeter polytopes;
- for many of the *mutation finite* cluster algebras there is an associated (quotient of a) Coxeter group ([BM], [FeTu2]);
- for *acyclic* cluster algebras there are associated actions of linear reflection groups.

There are many more (see for instance [FR], [SpT], [RSt], [RSp] and references therein) and probably, there should be even more yet to discover.

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