# Laguerre geometry in space forms and incircular nets

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**Abstract.** Classical (Euclidean) Laguerre geometry studies oriented hyperplanes, oriented hyperspheres, and their oriented contact in Euclidean space. We describe how this can be generalized to arbitrary Cayley-Klein spaces, in particular hyperbolic and elliptic space, and study the corresponding groups of Laguerre transformations. We give an introduction to Lie geometry and describe how these Laguerre geometries can be obtained as subgeometries. As an application of two-dimensional Lie and Laguerre geometry we study the properties of checkerboard incircular nets.

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# Contents

# 1 Introduction

Classically *Laguerre geometry* is the geometry of oriented hyperplanes, oriented hyperspheres, and their oriented contact in Euclidean space [Lag1885]. It is named after E. Laguerre [HPR1898], and was actively studied in dimensions 2 and 3 in the early twentieth century, see e.g. W. Blaschke [Bla1910, Bla1929]. More recently, Laguerre geometry has been employed to study, e.g. edge offset meshes and Laguerre minimal surfaces in the context of discrete differential geometry and architectural design [PGM2009, PGB2010, SPG2012].

The most comprehensive text on Laguerre geometry is the classical book by W. Blaschke [Bla1929], where however only the Euclidean case is treated. There exists no systematic presentation of non-Euclidean Laguerre geometry in the literature. The goal of the present paper is twofold. On one hand, it is supposed to be a comprehensive presentation of non-Euclidean Laguerre geometry, and thus has a character of a textbook. On the other hand, in Section 6 we demonstrate the power of Laguerre geometry on the example of the checkerboard incircular nets introduced in [AB2018]. We give a unified treatment of these nets in all space forms and describe them explicitly.

Generalizations of Laguerre geometry to non-Euclidean space have been studied by by H. Beck [Bec1910], U. Graf [Gra1934, Gra1937, Gra1939] and K. Fladt [Fla1956, Fla1957], mainly in dimension 2. We show how Laguerre geometry can be obtained in a unified way for arbitrary *Cayley-Klein spaces* [Kle1928, Gie1082] of any dimension. In the spirit of *Klein's Erlanger Programm* [Kle1893] this is done in a purely projective setup. The space of hyperplanes of a Cayley-Klein space $\mathcal{K} \subset \mathbb{R}\mathrm{P}^n$ is lifted to a quadric $\mathcal{B} \subset \mathbb{R}\mathrm{P}^{n+1}$, which we call the *Laguerre quadric*. Vice versa, the projection from the Laguerre quadric yields a double cover of the space of $\mathcal{K}$-hyperplanes which can be interpreted as carrying their orientation. In the projection hyperplanar sections of $\mathcal{B}$ correspond to spheres of the Cayley-Klein space $\mathcal{K}$. The corresponding group of quadric preserving transformations, which maps hyperplanar sections of $\mathcal{B}$ to hyperplanar sections of $\mathcal{B}$, naturally induces the group of transformations of oriented $\mathcal{K}$-hyperplanes, which preserves the oriented contact to Cayley-Klein spheres. We explicitly carry out this general construction in the cases of hyperbolic and elliptic geometry, yielding *hyperbolic Laguerre geometry* and *elliptic Laguerre geometry* respectively. The (classical) Euclidean case constitutes a degenerate case of this construction, which we handle in the appendix.

On the other hand the different Laguerre geometries appear as subgeometries of Lie geometry. *Lie (sphere) geometry* is the geometry of oriented hyperspheres of the $n$-sphere $\mathbb{S}^n$, and their oriented contact [Cec1992, Bla1929, BS2008]. Laguerre geometry can be obtained by distinguishing the set of "oriented hyperplanes" as a sphere complex among the set of oriented hyperspheres, the so called *plane complex*. Classically, the plane complex is taken to be parabolic, which leads to the notion of Euclidean Laguerre geometry, where elements of the plane complex are interpreted as oriented hyperplanes of Euclidean space [Bla1910, Bla1929, BS2008, Cec1992]. Choosing an elliptic or hyperbolic sphere complex, on the other hand, allows for the interpretation of the elements of the plane complex as oriented hyperplanes in hyperbolic or elliptic space. The group of *Lie transformations* that preserve the set of "oriented hyperplanes" covers the group of Laguerre transformations.

*Incircular nets* were introduced by W. Böhm [Böh1970] and are defined as congruences of straight lines in the plane with the combinatorics of the square grid such that each elementary quadrilateral admits an incircle. They are closely related to confocal conics. The construction and geometry of incircular nets and their Laguerre geometric generalization to *checkerboard incircular nets* have recently been discussed in great detail in [AB2018]. Explicit parametrizations for the Euclidean cases have been derived in [BST2018]. Higher dimensional analogues of incircular nets have been studied in [ABST2019]. We generalize planar checkerboard incircular nets to Lie geometry, which may be classified in terms of checkerboard incircular nets in hyperbolic/elliptic/Euclidean Laguerre geometry. We prove incidence theorems of Miquel type and show that all lines of a checkerboard incircular net are tangent to a *hypercycle*. This generalizes the results from [BST2018] and leads to a unified treatment of checkerboard incircular nets in all space forms.

The paper is structured in the following way:

In Section 1 we recall some basic notions from projective geometry, in particular on quadrics and polarity.

In Section 2 we introduce the notion of *Cayley-Klein spaces* and their corresponding groups of isometries. As an example, we give the projective models for hyperbolic and elliptic space. In sight of introducing the Laguerre geometries of those spaces we are particularly interested in the description of hyperplanes, spheres, and their mutual relations.

In Section 3 we study the general construction of *central projection* of a quadric from a point onto its polar hyperplane. This leads to a double cover of a Cayley-Klein space in the hyperplane such that the spheres in that Cayley-Klein space correspond to hyperplanar sections of the quadric. In this way, hyperbolic and elliptic geometry can be recovered as projections from *Möbius geometry*, and may be seen as the geometry of spheres in hyperbolic or elliptic space respectively. We demonstrate how the group of Möbius transformations can be decomposed into the corresponding isometries and scalings along concentric spheres.

In Section 4 we introduce the concept of *polar projection* of a quadric. Similar to the central projection of a quadric it yields a double cover of certain hyperplanes of a Cayley-Klein space. The double cover can be interpreted as carrying the orientation of those hyperplanes, and thus inducing a *Laguerre geometry*. We discuss the cases of hyperbolic and elliptic Laguerre geometry and their corresponding transformation groups, including coordinate representations for the different geometric objects appearing in each case. A decomposition into isometries and scalings can be obtained in an analogous way to the decomposition of the Möbius group.

In Section 5 we first give an elementary introduction into *Lie geometry*, which leads to its projective model. Afterwards we show how to unify Möbius and Laguerre geometry of Cayley-Klein spaces in the framework of Lie geometry by considering certain (compatible) sphere complexes. The group of Möbius transformations, Laguerre transformations, and isometries appear as quotients of the group of Lie transformations.

In Section 6, as an application of two-dimensional Lie and Laguerre geometry, we study the properties of *checkerboard incircular* nets in these geometries.

In the appendix we handle the (degenerate) Euclidean cases (see Section A), and study an invariant on a quadric induced by a point (see Section B). We show that the *signed inversive distance* is an example of the latter.

## 1.1 Projective geometry

Consider the $n$-dimensional *real projective space*

$$\mathbb{R}\mathrm{P}^n := \mathrm{P}(\mathbb{R}^{n+1}) := \left(\mathbb{R}^{n+1} \setminus \{0\}\right)\big/\sim$$

as it is generated via projectivization from its *homogeneous coordinate space* $\mathbb{R}^{n+1}$ using the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad x = \lambda y, \quad \text{for some } \lambda \in \mathbb{R}.$$

We denote points in $\mathbb{R}\mathrm{P}^n$ and its *homogeneous coordinates* by

$$\boldsymbol{x} = [x] = [x_1, \ldots, x_{n+1}].$$

*Affine coordinates* are given by normalizing one homogeneous coordinate to be equal to one and then dropping this coordinate, e.g.,

$$\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right).$$

Points with $x_{n+1} = 0$, for which this normalization is not possible, are said to lie on the hyperplane at infinity.

The *projectivization operator* P acts on any subset of the homogeneous coordinate space. In particular, a *projective subspace* $\boldsymbol{U} \subset \mathbb{R}\mathrm{P}^n$ is given by the projectivization of a linear subspace $U \subset \mathbb{R}^{n+1}$,

$$\boldsymbol{U} = \mathrm{P}(U), \quad \dim \boldsymbol{U} = \dim U - 1.$$

To denote projective subspaces spanned by a given set of points $\boldsymbol{a}_1, \ldots \boldsymbol{a}_k$ with linear independent homogeneous coordinate vectors we use the exterior product

$$\boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_k := [a_1 \wedge \cdots \wedge a_k] = \mathrm{P}(\mathrm{span}\{a_1, \ldots, a_k\}).$$

The group of *projective transformations* is induced by the group of linear transformations of $\mathbb{R}^{n+1}$ and denoted by $\mathrm{PGL}(n+1)$. A projective transformation maps projective subspaces to projective subspaces, while preserving their dimension and incidences. The *fundamental theorem of real projective geometry* states that this property characterizes projective transformations.

**Theorem 1.1.** *Let $n \geq 2$, and $W \subset \mathbb{R}\mathrm{P}^n$ an open subset. Let $f : W \to \mathbb{R}\mathrm{P}^n$ be an injective map that maps intersections of $k$-dimensional projective subspaces with $W$ to intersections of $k$-dimensional projective subspaces with $f(W)$ for some $1 \leq k \leq n-1$. Then $f$ is the restriction of a unique projective transformation of $\mathbb{R}\mathrm{P}^n$.*

For a projective subgroup $G \subset \mathrm{PGL}(n+1)$ we denote the *stabilizer* of a finite number of points $\boldsymbol{v}_1, \ldots \boldsymbol{v}_m \in RP^n$ by

$$G_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_m} := \{g \in G \mid g(\boldsymbol{v}_i) = \boldsymbol{v}_i, \text{ for } i = 1, \ldots, m\}. \tag{1}$$

## 1.2 Quadrics

Let $\langle \cdot, \cdot \rangle$ be a non-zero symmetric bilinear form on $\mathbb{R}^{n+1}$. A vector $x \in \mathbb{R}^{n+1}$ is called

- *spacelike* if $\langle x, x \rangle > 0$,
- *timelike* if $\langle x, x \rangle < 0$,
- *lightlike*, or *isotropic*, if $\langle x, x \rangle = 0$.

There always exists an *orthogonal basis* with respect to $\langle\cdot,\cdot\rangle$, i.e. a basis $(e_i)_{i=1,\dots,n+1}$ satisfying $\langle e_i, e_j\rangle = 0$ if $i \neq j$. The triple $(r,s,t)$, consisting of the numbers of spacelike $(r)$, timelike $(s)$, and lightlike $(t)$ vectors in $(e_i)_{i=1,\dots,n+1}$ is called the *signature* of $\langle\cdot,\cdot\rangle$. It is invariant under linear transformations. If $t=0$, the bilinear form $\langle\cdot,\cdot\rangle$ is called *non-degenerate*, in which case we might omit its value in the signature. We alternatively write the signature in the form

$$(\underbrace{+\cdots+}_{r}\underbrace{-\cdots-}_{s}\underbrace{0\cdots0}_{t}).$$

The space $\mathbb{R}^{n+1}$ together with a bilinear form of signature $(r,s,t)$ is denoted by $\mathbb{R}^{r,s,t}$. The zero set of the quadratic form corresponding to $\langle\cdot,\cdot\rangle$

$$\mathbb{L}^{r,s,t} := \left\{x \in \mathbb{R}^{n+1} \;\middle|\; \langle x,x\rangle = 0\right\}$$

is called the *light cone*, or *isotropic cone*. Its projectivization

$$\mathcal{Q} := \mathrm{P}(\mathbb{L}^{r,s,t})$$

defines a *quadric* in $\mathbb{R}\mathrm{P}^n$. The quadric $\mathcal{Q}$ separates the projective space $\mathbb{R}\mathrm{P}^n$ into two regions

$$\begin{aligned}\mathcal{Q}^+ &:= \{\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \mid \langle x,x\rangle > 0\}\,,\\ \mathcal{Q}^- &:= \{\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \mid \langle x,x\rangle < 0\}\,,\end{aligned} \tag{2}$$

which we call the two *sides* of the quadric.

The bilinear form corresponding to a quadric is well-defined up to a non-zero scalar multiple. Thus, the signature of a quadric is well-defined up to interchanging $r$ and $s$. The signature of a projective subspace $\boldsymbol{U} = \mathrm{P}(U)$ is defined by the signature of the bilinear form restricted to $U$. After a choice for the signs of the signature of $\mathcal{Q}$ the signs for the signature of $\boldsymbol{U}$ are fixed.

A projective subspace entirely contained in the quadric $\mathcal{Q}$ is called an *isotropic subspace*. A point $\boldsymbol{x} \in \mathcal{Q}$ contained in the kernel of the corresponding bilinear form $\langle\cdot,\cdot\rangle$, i.e.

$$\langle x,y\rangle = 0 \qquad \text{for all } y \in \mathbb{R}^{n+1}$$

is called a *vertex* of $\mathcal{Q}$. Only degenerate quadrics have vertices.

**Lemma 1.2.** *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric with signature $(r,s,t)$.*

*(i) If $\mathcal{Q}$ is degenerate, i.e. $t>0$, its set of vertices is a projective subspace of dimension $t-1$.*

*(ii) The quadric $\mathcal{Q}$ contains isotropic subspaces of dimension $\min\{r,s\}+t-1$ through every point.*

Consider the following examples of quadrics in $\mathbb{R}\mathrm{P}^n$ with different signatures.

**Example 1.1.**

(i) A quadric with signature $(n+1,0)$ is empty in $\mathbb{R}\mathrm{P}^n$. By either identifying the quadric with its bilinear form up to non-zero scalar multiples or by complexification $\mathcal{Q} \subset \mathbb{C}\mathrm{P}^n$, we consider this to be an admissible non-degenerate quadric, which only happens to have an empty real part. Note that one side of the quadric $\mathcal{Q}^+ = \mathbb{R}\mathrm{P}^n$ is the whole space, while the other side $\mathcal{Q}^- = \varnothing$ is empty.

(ii) A quadric with signature $(n,1)$ is an “oval quadric”. It is projectively equivalent to the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$.

(iii) A quadric with signature $(n-1,2)$ is a generalization of a doubly ruled quadric in $\mathbb{R}\mathrm{P}^3$. It contains lines as isotropic subspaces through every point, but no planes.

(iv) A quadric with signature $(r, s, 1)$ is a cone. It consists of all lines connecting its vertex to a non-degenerate quadric of signature $(r, s)$, given by its intersection with a hyperplane not containing the vertex. Note that if $r = 0$ or $s = 0$ (the real part of) the cone only consists of the vertex. The remaining part of the cone can be considered as imaginary (cf. Example (i)).

(v) A quadric with signature $(1, 0, n)$ is a "doubly counted" hyperplane.

For non-neutral signature, i.e. $rs \neq s$, and $s \neq 0$, the subgroup of projective transformations preserving the quadric $\mathcal{Q}$ is exactly the *projective orthogonal group* $\mathrm{PO}(r, s, t)$, i.e. the projectivization of all linear transformations that preserve the bilinear form $\langle \cdot, \cdot \rangle$.

*Remark* 1.1. In the case $r = s$ the statement remains true if we exclude projective transformations that interchange the two sides (2) of the quadric. In the case $rs = 0$ the statement remains true upon the identifications mentioned in Example 1.1 (i)).

For simplicity we call $\mathrm{PO}(r, s, t)$ the group of transformations that preserve the quadric $\mathcal{Q}$ for all signatures. The fundamental theorem of real projective geometry (see Theorem 1.1) may be specialized to quadrics.

**Theorem 1.3.** *Let $n \geq 3$, $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ a non-degenerate non-empty quadric in $\mathbb{R}\mathrm{P}^n$, and $W \subset \mathcal{Q}$ an open subset of the quadric. Let $f : W \to \mathcal{Q}$ be an injective map that maps intersections of $k$-dimensional projective subspaces with $W$ to intersections of $k$-dimensional projective subspaces with $f(W)$ for some $2 \leq k \leq n-1$. Then $f$ is the restriction of a unique projective transformation of $\mathbb{R}\mathrm{P}^n$ that preserves the quadric $\mathcal{Q}$.*

For a non-degenerate quadric every such transformation can be decomposed into a finite number of reflections in hyperplanes by the *theorem of Cartan and Dieudonné.*

**Theorem 1.4.** *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a non-degenerate quadric of signature $(r, s)$. Then each element of the corresponding projective orthogonal group $\mathrm{PO}(r, s)$ is the composition of at most $n + 1$ reflections in hyperplanes, i.e. transformations of the form*

$$\sigma_{\boldsymbol{q}} : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n, \qquad [x] \mapsto \left[ x - 2\frac{\langle x, q \rangle}{\langle q, q \rangle} q \right]$$

*for some $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$.*

## 1.3 Polarity

A quadric induces the notion of *polarity* between projective subspaces (see Figure 1). For a projective subspace $\boldsymbol{U} = \mathrm{P}(U) \subset \mathbb{R}\mathrm{P}^n$ , where $U \subset \mathbb{R}^{n+1}$ is a linear subspace, the *polar subspace* of $\boldsymbol{U}$ is defined as

$$\boldsymbol{U}^\perp := \mathrm{P}\left( \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \langle x, y \rangle = 0 \text{ for all } y \in U \right\} \right).$$

If $\mathcal{Q}$ is non-degenerate, the dimensions of two polar subspaces satisfy the following relation:

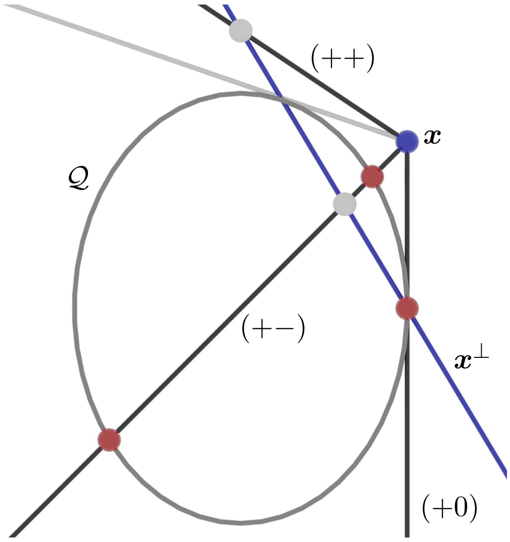
$$\dim \boldsymbol{U} + \dim \boldsymbol{U}^\perp = n - 1.$$

A refinement of this statement, which includes the signatures of the two polar subspaces, is captured in the following Lemma.

**Lemma 1.5.** *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a non-degenerate quadric of signature $(r, s)$. Then the signature $(\tilde{r}, \tilde{s}, \tilde{t})$ of a subspace $\boldsymbol{U} \subset \mathbb{R}\mathrm{P}^n$ and the signature $(\tilde{r}_\perp, \tilde{s}_\perp, \tilde{t}_\perp)$ of its polar subspace $\boldsymbol{U}^\perp$ with respect to $\mathcal{Q}$ satisfy*

$$r = \tilde{r} + \tilde{r}_\perp + \tilde{t}, \quad s = \tilde{s} + \tilde{s}_\perp + \tilde{t}, \quad \tilde{t} = \tilde{t}_\perp.$$

*In particular, $\tilde{t} \leq \min\{r, s\}$.*



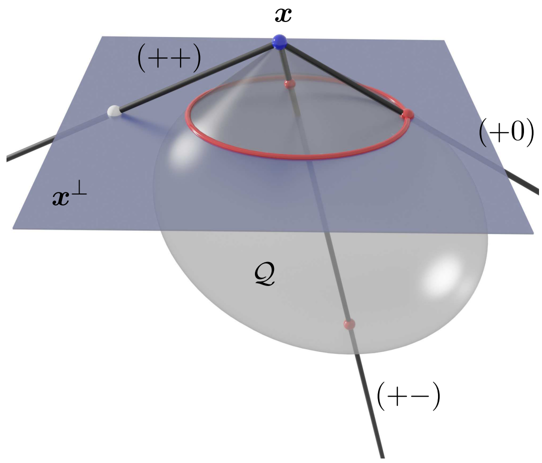




**Figure 1.** Polarity with respect to a conic $\mathcal{Q}$ with signature $(++-)$ in $\mathbb{RP}^2$ (*left*) and a quadric of signature $(+++-)$ in $\mathbb{RP}^3$ (*right*). The point $\boldsymbol{x}$ and its polar hyperplane $\boldsymbol{x}^\perp$ are shown as well as the cone of contact $C_{\mathcal{Q}}(\boldsymbol{x})$. Lines through $\boldsymbol{x}$ that are "inside" (signature $(+-)$), "on" (signature $(+0)$), and "outside" (signature $(++)$) the cone intersect the quadric in 2, 1, or 0 points respectively.

For a point $\boldsymbol{x} \in \mathcal{Q}$ on a quadric, which is not a vertex, the tangent hyperplane of $\mathcal{Q}$ at $\boldsymbol{x}$ is given by its polar hyperplane $\boldsymbol{x}^\perp$. If $\mathcal{Q}$ has signature $(r, s, t)$ then the tangent plane has signature $(r-1, s-1, t+1)$. Furthermore, for a non-degenerate quadric a projective subspace is tangent to $\mathcal{Q}$ if and only if its signature is degenerate.

A projective line not contained in a quadric can intersect the quadric in either zero, one, or two points (see Figure 1).

**Lemma 1.6.** *Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a quadric, and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{RP}^n$, $\boldsymbol{x} \neq \boldsymbol{y}$ be two points. Define*

$$\Delta := \langle x, y \rangle^2 - \langle x, x \rangle \, \langle y, y \rangle .$$

- *If $\Delta > 0$, then the line $\boldsymbol{x} \wedge \boldsymbol{y}$ has signature $(+-)$ and intersects $\mathcal{Q}$ in two points*

$$\boldsymbol{x}_\pm = \left[ \langle y, y \rangle \, x + \left( - \langle x, y \rangle \pm \sqrt{\Delta} \right) y \right].$$

- *If $\Delta < 0$, then the line $\boldsymbol{x} \wedge \boldsymbol{y}$ has signature $(++)$ or $(--)$ and intersects $\mathcal{Q}$ in no real points, but in two complex conjugate points*

$$\boldsymbol{x}_\pm = \left[ \langle y, y \rangle \, x + \left( - \langle x, y \rangle \pm i\sqrt{-\Delta} \right) y \right].$$

- *If $\Delta = 0$, then the line $\boldsymbol{x} \wedge \boldsymbol{y}$ has signature $(+0)$ or $(-0)$ and it is tangent to $\mathcal{Q}$ in the point*

$$\tilde{\boldsymbol{x}} = \left[ \langle y, y \rangle \, x - \langle x, y \rangle \, y \right],$$

*or it has signature $(00)$ and is contained in $\mathcal{Q}$*

The last point of the preceding lemma gives rise to the following definition of the cone of contact (see Figure 1).

**Definition 1.1.** Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a quadric with corresponding bilinear form $\langle \cdot, \cdot \rangle$, and $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$. Define the quadratic form

$$\Delta_x(y) := \langle x, y \rangle^2 - \langle x, x \rangle \, \langle y, y \rangle = 0$$

Then the corresponding quadric

$$C_{\mathcal{Q}}(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}\mathrm{P}^n \mid \Delta_x(y) = 0\}$$

is called the *cone of contact*, or tangent cone, to $\mathcal{Q}$ from the point $\boldsymbol{x}$.

The points of tangency of the cone of contact lie in the polar hyperplane of its vertex.

**Lemma 1.7.** *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric. For a point $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$ the cone of contact to $\mathcal{Q}$ from $\boldsymbol{x}$ is given by*

$$C_{\mathcal{Q}}(\boldsymbol{x}) = \bigcup_{\boldsymbol{y} \in \boldsymbol{x}^\perp \cap \mathcal{Q}} \boldsymbol{x} \wedge \boldsymbol{y}.$$

*Remark* 1.2. For a non-degenerate quadric $\mathcal{Q}$ the intersection $\boldsymbol{x}^\perp \cap \mathcal{Q}$ always results in a non-degenerate quadric in $\boldsymbol{x}^\perp$. If the restriction of the corresponding bilinear form has signature $(n, 0)$ or $(0, n)$ the intersection can be considered as imaginary. The real part of the cone only consists of the vertex in this case (cf. Example 1.1 (iv)).

## 1.4 Pencils of quadrics

Let $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathbb{R}\mathrm{P}^n$ be two distinct quadrics with corresponding bilinear forms $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ respectively. Every linear combination of these two bilinear forms yields a new quadric. The family of quadrics obtained by all linear combinations of the two bilinear forms is called a *pencil of quadrics* (see Figure 2):

$$\mathcal{Q}_1 \wedge \mathcal{Q}_2 := \left(\mathcal{Q}_{[\lambda_1, \lambda_2]}\right)_{[\lambda_1, \lambda_2] \in \mathbb{R}\mathrm{P}^1}, \qquad \mathcal{Q}_{[\lambda_1, \lambda_2]} := \{\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \mid \lambda_1 \langle x, x \rangle_1 + \lambda_2 \langle x, x \rangle_2 = 0\}.$$

It is a line in the *projective space of quadrics of* $\mathbb{R}\mathrm{P}^n$.

A pencil of quadrics is called *non-degenerate* if it does not consist exclusively of degenerate quadrics. It contains at most $n + 1$ degenerate quadrics.

A point contained in two quadrics from a pencil of quadrics is called a *base point*. It is then contained in every quadric of the pencil. The variety of base points has (at least) codimension 2.

**Example 1.2.** The pencil of quadrics $\mathcal{Q} \wedge C_{\mathcal{Q}}(\boldsymbol{x})$ spanned by a non-degenerate quadric $\mathcal{Q}$ and the cone of contact $C_{\mathcal{Q}}(\boldsymbol{x})$ from a point $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$ contains as degenerate quadrics only the cone $C_{\mathcal{Q}}(\boldsymbol{x})$ and the polar hyperplane $\boldsymbol{x}^\perp$. It is comprised of exactly the quadrics that are tangent to $\mathcal{Q}$ in $\mathcal{Q} \cap \boldsymbol{x}^\perp$.

# 2 Cayley-Klein spaces

Projective models for, e.g., hyperbolic, deSitter, and elliptic space can be obtained by using a quadric to induce the corresponding metric.

## 2.1 Cayley-Klein distance

A quadric $\mathcal{Q}$ within a projective space induces an invariant for pairs of points.

**Definition 2.1.** Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric with corresponding bilinear form $\langle \cdot, \cdot \rangle$. Then we denote by

$$K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) := \frac{\langle x, y \rangle^2}{\langle x, x \rangle \, \langle y, y \rangle}$$

the *Cayley-Klein distance* of any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$ that are not on the quadric.

In the presence of a Cayley-Klein distance the quadric $\mathcal{Q}$ is called the *absolute quadric.*

*Remark* 2.1. The name Cayley-Klein distance, or Cayley-Klein metric, is usually assigned to a metric derived from above quantity as, for example, the hyperbolic metric (cf. Section 2.4). Nevertheless, we prefer to assign it to this basic quantity associated with an arbitrary quadric.

The Cayley-Klein distance is projectively well-defined, in the sense that it depends neither on the choice of the bilinear form corresponding to the quadric $\mathcal{Q}$ nor on the choice of homogeneous coordinate vectors for the points $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, it is invariant under the group of projective transformations that preserve the quadric $\mathcal{Q}$, which we call the corresponding group of *isometries.*

The Cayley-Klein distance can be positive or negative depending on the relative location of the two points with respect to the quadric, cf. (2).

**Proposition 2.1.** *For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$ with $\langle x, y \rangle \neq 0$:*

- $K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) > 0$ *if $\boldsymbol{x}$ and $\boldsymbol{y}$ are on the same side of $\mathcal{Q}$,*
- $K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) < 0$ *if $\boldsymbol{x}$ and $\boldsymbol{y}$ are on opposite sides of $\mathcal{Q}$.*

A *Cayley-Klein space* is usually considered to be one side of the quadric, i.e. $\mathcal{Q}^+$ or $\mathcal{Q}^-$, together with a (pseudo-)metric derived from the Cayley-Klein distance, or equivalently, together with the transformation group of isometries.

## 2.2 Cayley-Klein spheres

Having a notion of "distance" allows for the definition of corresponding spheres (see Figure 2).

**Definition 2.2.** Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a quadric, $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$, and $\mu \in \mathbb{R} \cup \{\infty\}$. Then we call the set

$$S_\mu(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}\mathrm{P}^n \mid K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \mu\}$$

the *Cayley-Klein hypersphere* with *center* $\boldsymbol{x}$ and *Cayley-Klein radius* $\mu$.

*Remark* 2.2. Due to the fact that the Cayley-Klein sphere equation can be written as

$$\langle x, y \rangle^2 - \mu \langle x, x \rangle \, \langle y, y \rangle = 0 \tag{3}$$

we may include solutions on the quadric $\boldsymbol{y} \in \mathcal{Q}$ as well as allow for $\mu = \infty$ (cf. Proposition 2.3).

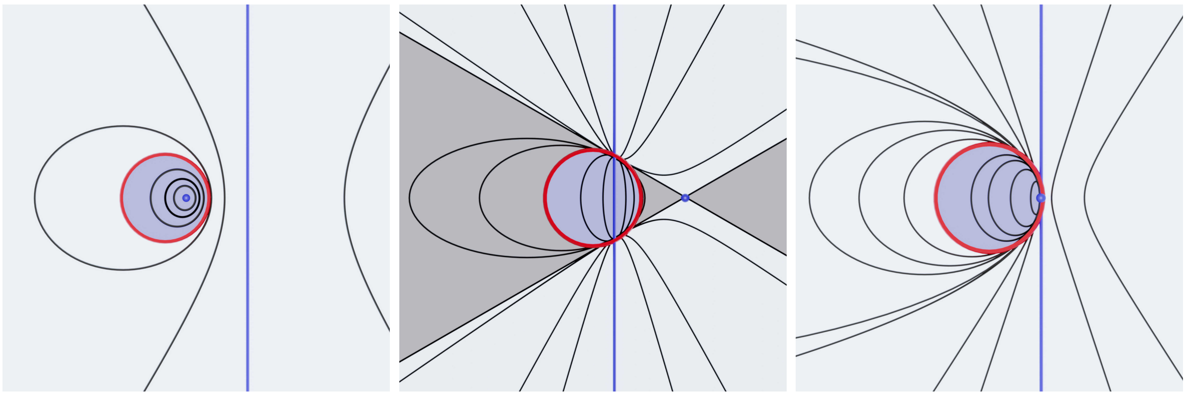


**Figure 2.** Concentric Cayley-Klein circles in the hyperbolic/deSitter plane. *Left*: Concentric circles with hyperbolic center. *Middle*: Concentric circles with deSitter center. *Right*: Concentric horocycles with center on the absolute conic.

*Remark* 2.3. Given the center $\boldsymbol{x}$ of a Cayley-Klein sphere one can further rewrite the Cayley-Klein sphere equation (3) as

$$\langle x, y\rangle^2 - \tilde{\mu}\,\langle y, y\rangle = 0,$$

where $\tilde{\mu} := \mu\,\langle x, x\rangle$. While $\tilde{\mu}$ is not projectively invariant anymore, the solution set of this equation still invariantly describes a Cayley-Klein sphere. We may now allow for centers on the quadric $\boldsymbol{x} \in \mathcal{Q}$ which gives rise to *Cayley-Klein horospheres* (see Figure 2, right).

**Proposition 2.2.** *For a Cayley-Klein sphere with center $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ and Cayley-Klein radius $\mu$ one has:*

- *If $\mu < 0$ the center and the points of a Cayley-Klein sphere are on opposite sides of the quadric.*
- *If $\mu > 0$ the center and the points of a Cayley-Klein sphere are on the same side of the quadric.*
- *If $\mu = 0$ the Cayley-Klein sphere is given by the (doubly counted)* ***polar hyperplane*** *$\boldsymbol{x}^\perp$.*
- *If $\mu = 1$ the Cayley-Klein sphere is the* ***cone of contact*** *$C_{\mathcal{Q}}(\boldsymbol{x})$ touching $\mathcal{Q}$, which is also called the* ***null-sphere*** *with center $\boldsymbol{x}$.*
- *If $\mu = \infty$ the Cayley-Klein sphere is the* ***absolute quadric*** *$\mathcal{Q}$.*

*Proof.* Follows from Proposition 2.1 and Lemma 1.7. □

Fixing the center and varying the radius of a Cayley-Klein sphere results in a family of concentric spheres (see Figure 2).

**Definition 2.3.** Given a quadric $\mathcal{Q} \subset \mathbb{RP}^n$ and a point $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ we call the family

$$(S_\mu(\boldsymbol{x}))_{\mu \in \mathbb{R} \cup \{\infty\}}$$

*concentric Cayley-Klein spheres* with center $\boldsymbol{x}$.

**Proposition 2.3.** *Let $\mathcal{Q} \subset \mathbb{RP}^n$ be a quadric. Then the family of concentric Cayley-Klein spheres with center $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ is the pencil of quadrics $\mathcal{Q} \wedge C_{\mathcal{Q}}(\boldsymbol{x})$ spanned by the absolute quadric $\mathcal{Q}$ and the cone of contact $C_{\mathcal{Q}}(\boldsymbol{x})$, or equivalently, by $\mathcal{Q}$ and the (doubly counted) polar hyperplane $\boldsymbol{x}^\perp$ (cf. Example 1.2).*

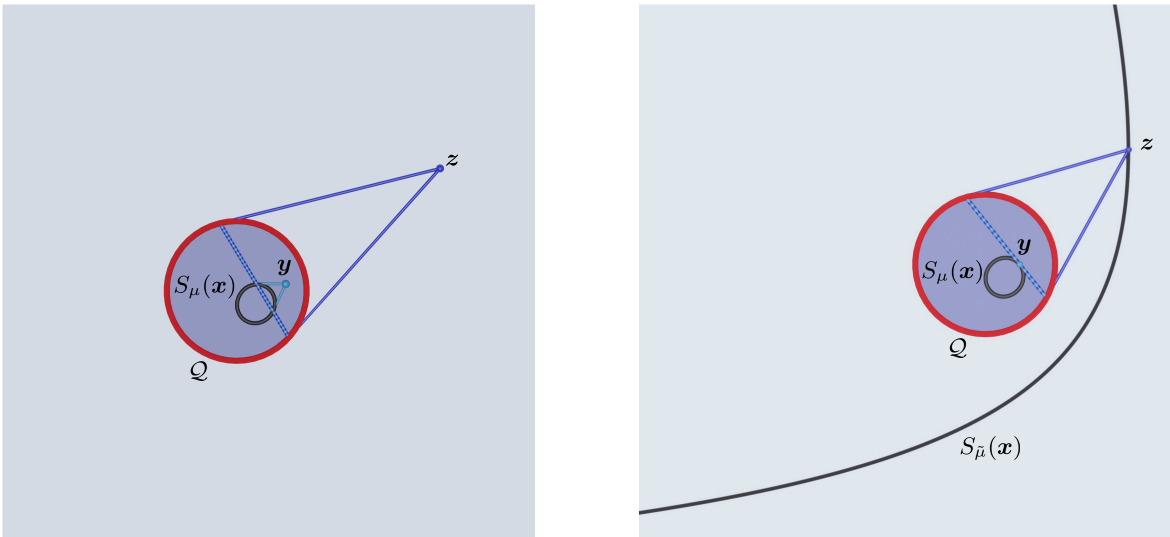



**Figure 3.** *Left*: Polarity with respect to a Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ and the absolute quadric $\mathcal{Q}$. *Right*: A Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ and its (concentric) polar Cayley-Klein sphere $S_{\tilde{\mu}}(\boldsymbol{x})$.

*Proof.* Writing the Cayley-Klein sphere equation as (3) we find that it is a linear equation in $\mu$ describing a pencil of quadrics. As observed in Proposition 2.2 it contains, in particular, the quadric $\mathcal{Q}$, the cone $C_{\mathcal{Q}}(\boldsymbol{x})$, and the hyperplane $\boldsymbol{x}^\perp$. □

This leads to a further characterization of Cayley-Klein spheres among all quadrics.

**Corollary 2.4.** *Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^n$ be a non-degenerate quadric. Then another quadric is a Cayley-Klein sphere if and only if it is tangent to $\mathcal{Q}$ in the (possibly imaginary) intersection with a hyperplane.*

*Proof.* Follows from Proposition 2.3 and Example 1.2. □

*Remark* 2.4. A pencil of concentric Cayley-Klein horospheres with center $\boldsymbol{x} \in \mathcal{Q}$ is spanned by the absolute quadric $\mathcal{Q}$ and the (doubly counted) tangent hyperplane $\boldsymbol{x}^\perp$, which yields third order contact between each horosphere and the absolute quadric.

### 2.3 Polarity of Cayley-Klein spheres

To describe spheres in terms of their tangent planes we turn our attention towards polarity in Cayley-Klein spheres (see Figure 3).

**Lemma 2.5.** *The bilinear form corresponding to a Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ with center $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n \setminus \mathcal{Q}$ and Cayley-Klein radius $\mu$ is given by*

$$b(y, \tilde{y}) = \langle x, y \rangle \langle x, \tilde{y} \rangle - \mu \langle x, x \rangle \langle y, \tilde{y} \rangle, \qquad y, \tilde{y} \in \mathbb{R}^{n+1}.$$

*Thus, for a point $\boldsymbol{y} \in \mathbb{R}\mathrm{P}^n$ the pole $\boldsymbol{z}$ with respect to the absolute quadric $\mathcal{Q}$ of the polar hyperplane of $\boldsymbol{y}$ with respect to $S_\mu(\boldsymbol{x})$ is given by*

$$z = \langle x, y \rangle x - \mu \langle x, x \rangle y. \tag{4}$$

*Proof.* The quadratic form of the Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ is given by (3):

$$\Delta(y) := \langle x, y \rangle^2 - \mu \langle x, x \rangle \langle y, y \rangle .$$

The corresponding bilinear form can be obtained by $b(y, \tilde{y}) = \frac{1}{2}(\Delta(y + \tilde{y}) - \Delta(y) - \Delta(\tilde{y}))$. □

For every point on a Cayley-Klein sphere the tangent hyperplane in that point is given by polarity in the Cayley-Klein sphere. Now the tangent hyperplanes of a Cayley-Klein sphere, in turn, may equivalently be described by their poles with respect to the absolute quadric $\mathcal{Q}$ (see Figure 3).

**Proposition 2.6.** *Let $\boldsymbol{x} \in \mathbb{RP}^n \setminus \mathcal{Q}$ and $\mu \in \mathbb{R} \setminus \{0, 1\}$. Then the poles (with respect to the absolute quadric $\mathcal{Q}$) of the tangent hyperplanes of the Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ are the points of a concentric Cayley-Klein sphere $S_{\tilde{\mu}}(\boldsymbol{x})$ with*

$$\mu + \tilde{\mu} = 1,$$

*and vice versa.*

*Proof.* Let $\boldsymbol{y} \in S_\mu(\boldsymbol{x})$ be a point on the Cayley-Klein sphere. Then the tangent plane to $S_\mu(\boldsymbol{x})$ at the point $\boldsymbol{y}$ is the polar plane of $\boldsymbol{y}$ with respect to $S_\mu(\boldsymbol{x})$. According to Lemma 2.5 the pole $\boldsymbol{z}$ of that tangent plane is given by (4). Computing the Cayley-Klein distance of this point to the center $\boldsymbol{x}$ we obtain

$$K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\langle x, z\rangle^2}{\langle x, x\rangle \ \langle z, z\rangle} = \frac{\langle x, y\rangle^2 \ (1-\mu)}{\langle x, y\rangle^2 \ (1-2\mu) + \mu^2 \ \langle x, x\rangle \ \langle y, y\rangle} = 1 - \mu,$$

where we used $\langle x, y\rangle^2 = \mu \ \langle x, x\rangle \ \langle y, y\rangle$. ☐

**Definition 2.4.** For a Cayley-Klein sphere $S_\mu(\boldsymbol{x})$ we call the Cayley-Klein sphere $S_{1-\mu}(\boldsymbol{x})$, consisting of all poles (with respect to the absolute quadric $\mathcal{Q}$) of tangent planes of $S_\mu(\boldsymbol{x})$, its *polar Cayley-Klein sphere.*

*Remark* 2.5. The two degenerate Cayley-Klein spheres $\boldsymbol{x}^\perp$ and $C_{\mathcal{Q}}(\boldsymbol{x})$ corresponding to the values $\mu = 0$ and $\mu = 1$ respectively, may be treated as being mutually polar. Then polarity defines a projective involution on a pencil of concentric Cayley-Klein spheres with fixed points at $\mu = \frac{1}{2}$ and $\mu = \infty$.

## 2.4 Hyperbolic geometry

Let $\langle\cdot,\cdot\rangle$ be the standard non-degenerate bilinear form of signature $(n, 1)$, i.e.

$$\langle x, y\rangle \coloneqq x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

for $x, y \in \mathbb{R}^{n+1}$, and denote by $\mathcal{S} \subset \mathbb{RP}^n$ the corresponding quadric. We identify the "inside" of $\mathcal{S}$, cf. (2), with the $n$-dimensional *hyperbolic space*

$$\mathcal{H} \coloneqq \mathcal{S}^-.$$

For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$ one has $K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}) \geq 1$, and the quantity $d$ given by

$$K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^2 d(\boldsymbol{x}, \boldsymbol{y})$$

defines a metric on $\mathcal{H}$ of constant negative sectional curvature. The corresponding group of isometries is given by $\mathrm{PO}(n, 1)$ and called the group of *hyperbolic motions.* The absolute quadric $\mathcal{S}$ consists of the points at (metric) infinity. We call the union

$$\overline{\mathcal{H}} \coloneqq \mathcal{H} \cup \mathcal{S}$$

the *compactified hyperbolic space.*

In this *projective model* of hyperbolic geometry *geodesics* are given by intersections of projective lines in $\mathbb{RP}^n$ with $\mathcal{H}$, while, more generally, *hyperbolic subspaces* (totally geodesic submanifolds) are given by intersections of projective subspaces in $\mathbb{RP}^n$ with $\mathcal{H}$. Thus, by polarity, every point $\boldsymbol{m} \in \mathrm{dS}$ in the "outside" of hyperbolic space,

$$\mathrm{dS} := \mathcal{S}^+,$$

which is called *deSitter space*, corresponds to a hyperbolic hyperplane $\boldsymbol{m}^\perp \cap \mathcal{H}$.

Consider two hyperbolic hyperplanes with poles $\boldsymbol{k}, \boldsymbol{m} \in \mathrm{dS}$.

- If $K_\mathcal{S}(\boldsymbol{k}, \boldsymbol{m}) < 1$, the two hyperplanes intersect in $\mathcal{H}$, and their hyperbolic intersection angle $\alpha$, or equivalently its conjugate angle $\pi - \alpha$ is given by

$$K_\mathcal{S}(\boldsymbol{k}, \boldsymbol{m}) = \cos^2 \alpha(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp).$$

- If $K_\mathcal{S}(\boldsymbol{k}, \boldsymbol{m}) > 1$, the two hyperplanes do not intersect in $\mathcal{H}$, and their hyperbolic distance is given by

$$K_\mathcal{S}(\boldsymbol{k}, \boldsymbol{m}) = \cosh^2 d(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp).$$

  The corresponding projective hyperplanes intersect in $(\boldsymbol{k} \wedge \boldsymbol{m})^\perp \subset \mathrm{dS}$.

- If $K_\mathcal{S}(\boldsymbol{k}, \boldsymbol{m}) = 0$, the two hyperplanes are parallel, i.e., they intersect on $\mathcal{S}$.

Finally, the hyperbolic distance of a point $\boldsymbol{x} \in \mathcal{H}$ and a hyperbolic hyperplane with pole $\boldsymbol{m} \in \mathrm{dS}$ is given by

$$K_\mathcal{S}(\boldsymbol{x}, \boldsymbol{m}) = -\sinh^2 d(\boldsymbol{x}, \boldsymbol{m}^\perp).$$

It is occasionally useful to employ a certain normalization of the homogeneous coordinate vectors:

$$\begin{aligned}
\mathbb{H}^n &:= \left\{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n,1} \;\middle|\; \langle x, x \rangle = -1,\ x_{n+1} \geq 0 \right\}, \\
\widetilde{\mathrm{d}\mathbb{S}}^n &:= \left\{ m = (m_1, \ldots, m_{n+1}) \in \mathbb{R}^{n,1} \;\middle|\; \langle m, m \rangle = 1 \right\}.
\end{aligned}$$

Then $\mathrm{P}(\mathbb{H}^n) = \mathcal{H}$ is an embedding and $\mathrm{P}(\widetilde{\mathrm{d}\mathbb{S}}^n) = \mathrm{dS}$ is a double cover. For $x, y \in \mathbb{H}^n$ and $k, m \in \widetilde{\mathrm{d}\mathbb{S}}^n$ above distance formulas become

$$\begin{aligned}
\langle x, y \rangle &= -\cosh d(\boldsymbol{x}, \boldsymbol{y}), \\
|\langle k, m \rangle| &= \cos \alpha(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp), \quad \text{if } |\langle k, m \rangle| \leq 1 \\
|\langle k, m \rangle| &= \cosh d(\boldsymbol{k}^\perp, \boldsymbol{m}^\perp), \quad \text{if } |\langle k, m \rangle| \geq 1 \\
|\langle x, m \rangle| &= \sinh d(\boldsymbol{x}, \boldsymbol{m}^\perp).
\end{aligned}$$

*Remark* 2.6. The double cover $\mathrm{P}(\widetilde{\mathrm{d}\mathbb{S}}^n) = \mathrm{dS}$ of deSitter space can be used to encode the orientation of the corresponding polar hyperplanes, e.g., by endowing the hyperbolic hyperplane corresponding to $m \in \widetilde{\mathrm{d}\mathbb{S}}^n$ with a normal in the direction of the hyperbolic halfspace on which the bilinear form with points $x \in \mathbb{H}^n$ is positive: $\langle x, m \rangle > 0$. Using the double cover to encode orientation one may omit the absolute value in $\langle x, m \rangle = \cos d$ to obtain an oriented hyperbolic distance $d$ between a point and an hyperbolic hyperplane. Similarly, one may omit the absolute value in $\langle k, m \rangle = \cos \alpha$ which allows to distinguish the intersection angle $\alpha$ and its conjugate angle $\pi - \alpha$.

We now turn our attention to the Cayley-Klein spheres of hyperbolic/deSitter geometry. First, consider a pencil of concentric Cayley-Klein spheres $S_\mu(\boldsymbol{x})$ with center inside hyperbolic space $\boldsymbol{x} \in \mathcal{H}$, $x \in \mathbb{H}^n$. Depending on the value of $\mu \in \mathbb{R} \cup \{\infty\}$ we obtain the following types of hyperbolic/deSitter spheres (see Figure 2, left):

- $\mu < 0$: A *deSitter sphere* with hyperbolic center.

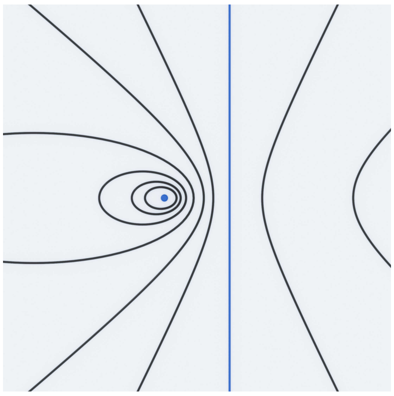


**Figure 4.** Concentric Cayley-Klein circles in the elliptic plane.

- $0 < \mu < 1$: $S_\mu(\boldsymbol{x})$ is empty.
- $1 < \mu < \infty$: A *hyperbolic sphere* with center $\boldsymbol{x} \in \mathcal{H}$ and hyperbolic radius $r = \operatorname{arcosh}\sqrt{\mu} > 0$:

$$S_\mu(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathcal{H} \;\middle|\; K_{\mathcal{S}}(\boldsymbol{x}, \boldsymbol{y}) = \cosh^2 r \right\} = \mathrm{P}\left(\{ y \in \mathbb{H}^n \mid \langle x, y \rangle = -\cosh r \}\right).$$

Second, consider a pencil of concentric Cayley-Klein spheres $S_\mu(\boldsymbol{m})$ with center outside hyperbolic space $\boldsymbol{m} \in \mathrm{dS}$, $m \in \widetilde{\mathrm{d}\mathbb{S}}^n$ (see Figure 2, middle):

- $\mu < 0$: A *hypersurface of constant hyperbolic distance* $r = \operatorname{arsinh}\sqrt{\mu} > 0$ to the hyperbolic plane $\boldsymbol{m}^\perp \cap \mathcal{H}$:

$$S_\mu(\boldsymbol{m}) = \left\{ \boldsymbol{y} \in \mathcal{H} \;\middle|\; K_{\mathcal{S}}(\boldsymbol{m}, \boldsymbol{y}) = -\sinh^2 r \right\} = \mathrm{P}\left(\left\{ y \in \mathbb{H}^N \;\middle|\; |\langle m, y \rangle| = \sinh r \right\}\right).$$

- $0 < \mu < 1$: A *deSitter sphere* tangent to $\mathcal{S}$. All its tangent hyperplanes are hyperbolic hyperplanes.
- $1 < \mu < \infty$: A *deSitter sphere* tangent to $\mathcal{S}$ with no hyperbolic tangent hyperplanes.

Third, a pencil of concentric Cayley-Klein horospheres with center on the absolute quadric $\boldsymbol{x} \in \mathcal{S}$, $x \in \mathbb{L}^{n,1}$ consists of *hyperbolic horospheres* and *deSitter horospheres* (see Figure 2, right).

### 2.5 Elliptic geometry

For $x, y \in \mathbb{R}^{n+1}$ we denote by

$$x \cdot y := x_1 y_1 + \ldots x_n y_n + x_{n+1} y_{n+1}$$

the standard (positive definite) scalar product on $\mathbb{R}^{n+1}$, i.e. the standard non-degenerate bilinear form of signature $(n+1, 0)$. The corresponding quadric $\mathcal{O} \subset \mathbb{RP}^n$ is empty (or purely imaginary, cf. Example 1.1 (i)), as well as the set $\mathcal{O}^- = \varnothing$, while

$$\mathcal{E} := \mathcal{O}^+ = \mathbb{RP}^n$$

is the whole projective space, which we identify with the $n$-dimensional *elliptic space*. For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$ one always has $0 \le K_{\mathcal{O}}(\boldsymbol{x}, \boldsymbol{y}) \le 1$ and the quantity $d$ given by

$$K_{\mathcal{O}}(\boldsymbol{x}, \boldsymbol{y}) = \cos^2 d(\boldsymbol{x}, \boldsymbol{y})$$

defines a metric on $\mathcal{E}$ of constant positive sectional curvature. The corresponding group of isometries is given by $\mathrm{PO}(n+1)$ and called the group of *elliptic motions*.

In this *projective model* of elliptic geometry *geodesics* are given by projective lines, while, more generally, *elliptic subspaces* are given by projective subspaces. By polarity, there is a one-to-one correspondence of points $\boldsymbol{x} \in \mathcal{E}$ in elliptic space and elliptic hyperplanes $\boldsymbol{x}^{\perp}$.

Two hyperplanes in elliptic space always intersect. If $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$ are the poles of two elliptic hyperplanes, then their intersection angle $\alpha$, or equivalently its conjugate angle $\pi - \alpha$ is given by

$$K_{\mathcal{O}}\left(\boldsymbol{x}, \boldsymbol{y}\right) = \cos^2 \alpha(\boldsymbol{x}^{\perp}, \boldsymbol{y}^{\perp}).$$

The distance of a point $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^n$ and an elliptic hyperplane with pole $\boldsymbol{y} \in \mathbb{R}\mathrm{P}^n$ is given by

$$K_{\mathcal{O}}\left(\boldsymbol{x}, \boldsymbol{y}\right) = \sin^2 \alpha(\boldsymbol{x}, \boldsymbol{y}^{\perp}).$$

One may normalize the homogeneous coordinate vectors of points in elliptic space to lie on a sphere:

$$\mathbb{S}^n := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; x \cdot x = 1 \right\}.$$

Then $\mathrm{P}(\mathbb{S}^n) = \mathcal{E}$ is a double cover, where antipodal points of the sphere are identified. In this normalization elliptic planes correspond to great spheres of $\mathbb{S}^n$, and it turns out that elliptic geometry is a double cover of *spherical geometry.* For $x, y \in \mathbb{S}^n$ above distance formulas become

$$\begin{aligned}
|x \cdot y| &= \cos d(\boldsymbol{x}, \boldsymbol{y}), \\
|x \cdot y| &= \cos \alpha(\boldsymbol{x}^{\perp}, \boldsymbol{y}^{\perp}), \\
|x \cdot y| &= \sin d(\boldsymbol{x}, \boldsymbol{y}^{\perp}),
\end{aligned}$$

*Remark* 2.7. The pole $\boldsymbol{x} \in \mathcal{E}$ of an elliptic hyperplane $\boldsymbol{x}^{\perp}$ has two lifts to the sphere, $x, -x \in \mathbb{S}^n$, which may be used to encode the orientation of the hyperplane (cf. Remark 2.6). This allows for omitting the absolute values is above distance formulas, while taking distances to be signed and distinguishing between intersection angles and their conjugate angles.

A Cayley-Klein sphere in elliptic space $S_\mu(\boldsymbol{x})$ with center $\boldsymbol{x} \in \mathcal{E}$, $x \in \mathbb{S}^n$, is not empty if and only if $0 \leq \mu \leq 1$ (see Figure 4). In this case it corresponds to an *elliptic sphere* with center $\boldsymbol{x} \in \mathcal{E}$ and elliptic radius $0 \leq r = \arccos \sqrt{\mu} \leq \frac{\pi}{2}$:

$$S_\mu(\boldsymbol{x}) = \left\{ \boldsymbol{y} \in \mathcal{E} \;\middle|\; K_{\mathcal{O}}\left(\boldsymbol{x}, \boldsymbol{y}\right) = \cos^2 r \right\} = \mathrm{P}\left( \left\{ y \in \mathbb{S}^n \mid |x \cdot y| = \cos r \right\} \right).$$

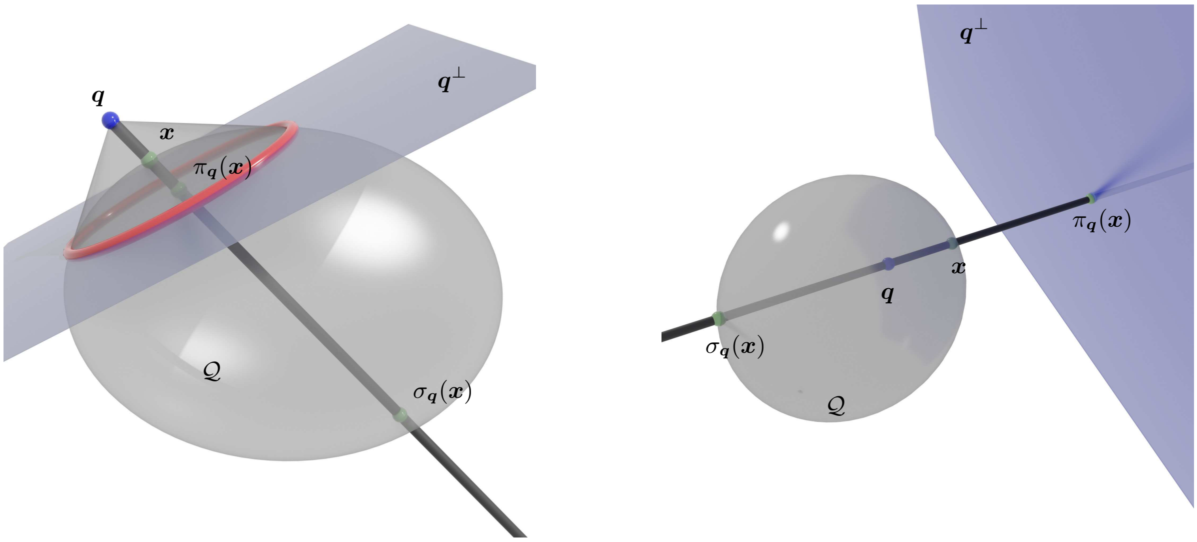



**Figure 5.** The involution and projection of an oval quadric $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^3$ induced by a point $\boldsymbol{q}$ not on the quadric. *L*eft: The point $\boldsymbol{q}$ lies "outside" the quadric. *R*ight: The point $\boldsymbol{q}$ lies "inside" the quadric.

# 3 Central projection of quadrics

The Cayley-Klein spaces introduced in Section 2 can be lifted to a quadric in a projective space of one dimension higher, such that Cayley-Klein spheres lift to hyperplanar sections of the quadric. Vice versa, the central projection of a quadric from a point yields a double cover of a Cayley-Klein space in one dimension less. In this way, e.g., hyperbolic and elliptic geometry can be obtained from Möbius geometry.

## 3.1 The involution and projection induced by a point

Let $\langle\cdot,\cdot\rangle$ be a bilinear form on $\mathbb{R}^{n+2}$ of signature $(r,s,t)$, and denote by $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^{n+1}$ the corresponding quadric. We are concerned with the central projection of $\mathcal{Q}$ from a point $\boldsymbol{q}$ not on the quadric onto a hyperplane of $\mathbb{R}\mathrm{P}^{n+1}$ which is canonically chosen to be the polar plane of $\boldsymbol{q}$.

**Definition 3.1.** A point $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+1} \setminus \mathcal{Q}$ not on the quadric induces two maps

$$\sigma_{\boldsymbol{q}}, \pi_{\boldsymbol{q}} \colon \mathbb{R}\mathrm{P}^{n+1} \to \mathbb{R}\mathrm{P}^{n+1}$$

$$\sigma_{\boldsymbol{q}} : [x] \mapsto [\sigma_q(x)] = \left[x - 2\frac{\langle x,q\rangle}{\langle q,q\rangle}q\right], \qquad \pi_{\boldsymbol{q}} : [x] \mapsto [\pi_q(x)] = \left[x - \frac{\langle x,q\rangle}{\langle q,q\rangle}q\right],$$

which we call the *associated involution* and *projection* respectively.

*Remark* 3.1. The involution $\sigma_{\boldsymbol{q}}$ is also called *reflection in the hyperplane* $\boldsymbol{q}^\perp$ (cf. Theorem 1.4).

We summarize the main properties of the involution and projection associated with the point $\boldsymbol{q}$ in the following proposition.

**Proposition 3.1.**

*(i) The map $\sigma_{\boldsymbol{q}}$ is a projective involution that fixes $\boldsymbol{q}$, i.e.,*

$$\sigma_{\boldsymbol{q}} \in \mathrm{PO}(r,s,t)_{\boldsymbol{q}}, \qquad \sigma_{\boldsymbol{q}} \circ \sigma_{\boldsymbol{q}} = \mathrm{id},$$

*It further fixes every point on the polar hyperplane $\boldsymbol{q}^\perp$.*

*For every line through $\boldsymbol{q}$ that intersects the quadric $\mathcal{Q}$ the involution $\sigma_{\boldsymbol{q}}$ interchanges the two intersection points, while for a line through $\boldsymbol{q}$ that touches the quadric $\mathcal{Q}$ it fixes the touching point (cf. Lemma 1.6).*

*(ii) The map $\pi_{\boldsymbol{q}}$ is a projection onto $\boldsymbol{q}^\perp \simeq \mathbb{RP}^n$. Its restriction onto the quadric*

$$\pi_{\boldsymbol{q}}\big|_{\mathcal{Q}} : \mathcal{Q} \to \pi_{\boldsymbol{q}}(\mathcal{Q})$$

*is a double cover with branch locus $\mathcal{Q} \cap \boldsymbol{q}^\perp$.*

*(iii) The involution and projection together satisfy*

$$\pi_{\boldsymbol{q}} \circ \sigma_{\boldsymbol{q}} = \pi_{\boldsymbol{q}}.$$

*Vice versa, if two distinct points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{RP}^{n+1}$ project to the same point $\pi_{\boldsymbol{q}}(\boldsymbol{x}) = \pi_{\boldsymbol{q}}(\boldsymbol{y})$, then $\boldsymbol{x} = \sigma_{\boldsymbol{q}}(\boldsymbol{y})$. This gives rise to a one-to-one correspondence of the projection and the quotient*

$$\pi_{\boldsymbol{q}}(\mathcal{Q}) \simeq \mathcal{Q}/\sigma_{\boldsymbol{q}}.$$

*Remark* 3.2. The involution $\sigma_{\boldsymbol{q}}$ and projection $\pi_{\boldsymbol{q}}$ act in the same way as described in Proposition 3.1 on every quadric from the pencil $\mathcal{Q} \wedge C_{\mathcal{Q}}(\boldsymbol{q})$ spanned by $\mathcal{Q}$ and the cone of contact $C_{\mathcal{Q}}(\boldsymbol{q})$ with vertex $\boldsymbol{q}$ (cf. Example 1.2).

The intersection

$$\widetilde{\mathcal{Q}} := \mathcal{Q} \cap \boldsymbol{q}^\perp$$

is a quadric of signature

- $(r-1, s, t)$ if $\langle q, q \rangle > 0$, or
- $(r, s-1, t)$ if $\langle q, q \rangle < 0$.

The projection of a quadric $\mathcal{Q} \subset \mathbb{RP}^{n+1}$ from a point $\boldsymbol{q} \in \mathbb{RP}^{n+1} \setminus \mathcal{Q}$ onto its polar hyperplane $\boldsymbol{q}^\perp$ is a double cover of the "inside" or the "outside", cf. (2), of $\widetilde{\mathcal{Q}} = \boldsymbol{q}^\perp \cap \mathcal{Q}$ depending on the signature of $\boldsymbol{q}$.

**Proposition 3.2.** *Let $\boldsymbol{q} \in \mathbb{RP}^{n+1} \setminus \mathcal{Q}$. Then*

- $\pi_{\boldsymbol{q}}(\mathcal{Q}) = \widetilde{\mathcal{Q}}^- \cup \widetilde{\mathcal{Q}}$, *if* $\langle q, q \rangle > 0$,
- $\pi_{\boldsymbol{q}}(\mathcal{Q}) = \widetilde{\mathcal{Q}}^+ \cup \widetilde{\mathcal{Q}}$, *if* $\langle q, q \rangle < 0$,

*Proof.* Decompose the homogeneous coordinate vector of a point $\boldsymbol{x} \in \mathcal{Q}$ into its projection onto $q$ and $q^\perp$

$$x = \alpha q + \pi_q(x),$$

with some $\alpha \in \mathbb{R}$. Then

$$0 = \langle x, x \rangle = \alpha^2 \langle q, q \rangle + \langle \pi_q(x), \pi_q(x) \rangle$$

and thus

$$\langle \pi_q(x), \pi_q(x) \rangle = -\alpha^2 \langle q, q \rangle \begin{cases} < 0, & \text{if } \langle q, q \rangle \geq 0 \\ > 0, & \text{if } \langle q, q \rangle \leq 0. \end{cases}$$

□

The following proposition shows how the Cayley-Klein distance induced by $\widetilde{\mathcal{Q}}$ for points in the projection $\pi_{\boldsymbol{q}}(\mathcal{Q})$ can be lifted to the points on $\mathcal{Q}$.

**Proposition 3.3.** *Let $\boldsymbol{q} \in \mathbb{RP}^{n+1} \setminus \mathcal{Q}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{Q}$. Then the Cayley-Klein distance with respect to $\widetilde{\mathcal{Q}}$ of their projections $\pi_{\boldsymbol{q}}(\boldsymbol{x}), \pi_{\boldsymbol{q}}(\boldsymbol{y})$ is given by*

$$K_{\widetilde{\mathcal{Q}}}(\pi_{\boldsymbol{q}}(\boldsymbol{x}), \pi_{\boldsymbol{q}}(\boldsymbol{y})) = \left(1 - \frac{\langle x, y \rangle \langle q, q \rangle}{\langle x, q \rangle \langle y, q \rangle}\right)^2. \tag{5}$$

*Proof.* We decompose the homogeneous coordinate vectors of $x, y$ into their projections onto $q$ and $q^\perp$

$$x = \alpha q + \pi_q(x), \qquad y = \beta q + \pi_q(y)$$

with some $\alpha, \beta \in \mathbb{R}$. Then,

$$1 - \frac{\langle x, y\rangle \, \langle q, q\rangle}{\langle x, q\rangle \, \langle y, q\rangle} = 1 - \frac{\Big(\alpha\beta \, \langle q, q\rangle + \langle x, y\rangle_q\Big) \, \langle q, q\rangle}{\alpha\beta \, \langle q, q\rangle^2} = -\frac{\langle \pi_q(x), \pi_q(y)\rangle}{\alpha\beta \, \langle q, q\rangle}.$$

Now with

$$0 = \langle x, x\rangle = \alpha^2 \, \langle q, q\rangle + \langle \pi_q(x), \pi_q(x)\rangle \, ,$$

and the analogous equation for $y$ we obtain

$$\frac{\langle \pi_q(x), \pi_q(y)\rangle^2}{\alpha^2\beta^2 \, \langle q, q\rangle^2} = \frac{\langle \pi_q(x), \pi_q(y)\rangle^2}{\langle \pi_q(x), \pi_q(x)\rangle \, \langle \pi_q(y), \pi_q(y)\rangle}.$$

□

*Remark* 3.3. Omitting the square for the quantity on the right hand side of equation (5) leads to a signed version of the lifted Cayley-Klein distance (see Section B).

While the Cayley-Klein distance can, in general, be both positive or negative, the right hand side of equation (5) is always positive. This corresponds to the fact that the projection of $\mathcal{Q}$ only always covers one side of $\widetilde{\mathcal{Q}}$. Though having no real preimages the points on the other side of $\widetilde{\mathcal{Q}}$ may be viewed as projections of certain imaginary points of $\mathcal{Q}$ (see Proposition B.4).

The transformation group induced by $\mathrm{PO}(r,s,t)_{\boldsymbol{q}}$, cf. (1), onto $\boldsymbol{q}^\perp$ is exactly the group of projective transformations $\mathrm{PO}(\tilde{r}, \tilde{s}, \tilde{t})$ that preserve the quadric $\widetilde{\mathcal{Q}}$. It is doubly covered by $\mathrm{PO}(r,s,t)_{\boldsymbol{q}}$ and can be identified with the quotient

$$\mathrm{PO}(\tilde{r}, \tilde{s}, \tilde{t}) \simeq {}^{\mathrm{PO}(r,s,t)_{\boldsymbol{q}}}\!\big/_{\sigma_{\boldsymbol{q}}}.$$

Note that $\mathrm{PO}(r,s,t)_{\boldsymbol{q}}$ is the largest subgroup of $\mathrm{PO}(r,s,t)$ admitting this quotient, i.e. the subgroup of transformations that commute with $\sigma_{\boldsymbol{q}}$.

### 3.2 Cayley-Klein spheres as planar sections

From now on, let $\mathcal{Q}$ be a non-degenerate quadric of signature $(r,s)$. Then each sections of the quadric $\mathcal{Q}$ with a hyperplane can be identified with the pole of the corresponding hyperplane.

**Definition 3.2.** We call a non-empty intersection of the quadric $\mathcal{Q}$ with a hyperplane a *$\mathcal{Q}$-spheres*, and identify it with the pole of the hyperplane. Thus, we call

$$\mathfrak{S} := \left\{ \boldsymbol{x} \in \mathbb{R}\mathrm{P}^{n+1} \;\middle|\; \boldsymbol{x}^\perp \cap \mathcal{Q} \neq \varnothing \right\}$$

the space of $\mathcal{Q}$-spheres.

*Remark* 3.4. Depending on the signature of of the quadric $\mathcal{Q}$ there are only three possible cases, assuming, w.l.o.g., $r \geq s$:

- $\mathfrak{S} = \varnothing$ if $\mathcal{Q}$ has signature $(n+2, 0)$,
- $\mathfrak{S} = \mathcal{Q}^+ \cup \mathcal{Q}$ if $\mathcal{Q}$ has signature $(n+1, 1)$,
- $\mathfrak{S} = \mathbb{R}\mathrm{P}^{n+1}$ else.

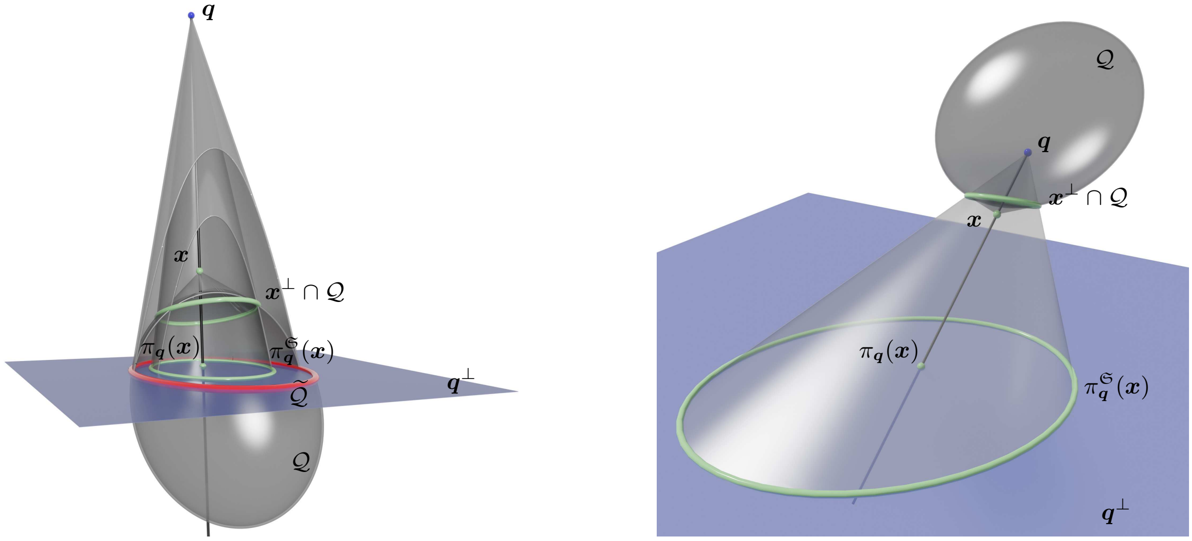



**Figure 6.** The central projection of a hyperplanar section $\boldsymbol{x}^\perp \cap \mathcal{Q}$ of a quadric $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^3$ with respect to a point $\boldsymbol{q}$. Its image is a Cayley-Klein sphere $\pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x}) \subset \pi_{\boldsymbol{q}}(\mathcal{Q})$ with respect to the absolute quadric $\widetilde{\mathcal{Q}}$. Its center is given by $\pi_{\boldsymbol{q}}(\boldsymbol{x})$. The cone of contact can be used to distinguish the type of Cayley-Klein sphere that is obtained in the projection.

It turns out that every $\mathcal{Q}$-sphere projects down to a Cayley-Klein sphere in $\pi_{\boldsymbol{q}}(\mathcal{Q})$, where the type of sphere can be distinguished by the two sides of the cone of contact $C_{\mathcal{Q}}(\boldsymbol{q})$. Denote by

$$\Delta_q(x) = \langle x, q\rangle^2 - \langle x, x\rangle \langle q, q\rangle = -\langle q, q\rangle \langle \pi_q(x), \pi_q(x)\rangle \tag{6}$$

the quadratic form of the cone of contact $C_{\mathcal{Q}}(\boldsymbol{q})$ (see Definition 1.1).

**Proposition 3.4.** *Consider the map*

$$\pi_{\boldsymbol{q}}^{\mathfrak{S}} : \boldsymbol{x} \mapsto \pi_{\boldsymbol{q}}(\boldsymbol{x}^\perp \cap \mathcal{Q}),$$

*for $\boldsymbol{x} \in \mathfrak{S}$. Then for every $\boldsymbol{x} \in \mathfrak{S}$ the image $\pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x})$ is a Cayley-Klein sphere or horosphere with points in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ (see Figure 6).*

- ▶ *For $\Delta_q(x) \neq 0$ the image is a Cayley-Klein sphere with center $\pi_{\boldsymbol{q}}(\boldsymbol{x})$ and Cayley-Klein radius*

$$\mu = \frac{\langle x, q\rangle^2}{\Delta_q(x)},$$

*i.e.*

$$\pi_{\boldsymbol{q}}(\boldsymbol{x}^\perp \cap \mathcal{Q}) = S_\mu\left(\pi_{\boldsymbol{q}}(\boldsymbol{x})\right).$$

  - *If $\Delta_q(x) > 0$, then $\pi_{\boldsymbol{q}}(\boldsymbol{x}) \in \pi_{\boldsymbol{q}}(\mathcal{Q}) \setminus \widetilde{\mathcal{Q}}$.*
  - *If $\Delta_q(x) < 0$, then $\pi_{\boldsymbol{q}}(\boldsymbol{x}) \in \boldsymbol{q}^\perp \setminus \pi_{\boldsymbol{q}}(\mathcal{Q})$.*

- ▶ *For $\Delta_q(x) = 0$ the image is a Cayley-Klein horosphere with center $\pi_{\boldsymbol{q}}(x) \in \widetilde{\mathcal{Q}}$.*
- ▶ *For $\boldsymbol{x} \in \mathfrak{S} \cap \boldsymbol{q}^\perp$ the image is a hyperplane in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ with pole $\boldsymbol{x}$.*
- ▶ *For $\boldsymbol{x} \in \mathcal{Q}$ the image is the cone of contact $C_{\widetilde{\mathcal{Q}}}(\pi_{\boldsymbol{q}}(\boldsymbol{x})) \subset \pi_{\boldsymbol{q}}(\mathcal{Q})$.*
- ▶ *For $\boldsymbol{x} = \boldsymbol{q}$ the image is the absolute quadric $\widetilde{\mathcal{Q}}$.*

*Proof.* We show the claim for points not on the cone of contact. Thus, let $\boldsymbol{x} \in \mathfrak{S} \setminus C_{\mathcal{Q}}(\boldsymbol{q})$, i.e., $\Delta_q(x) \neq 0$. Let $\boldsymbol{y} \in \boldsymbol{x}^\perp \cap \mathcal{Q}$ be a point on the corresponding $\mathcal{Q}$-sphere. Then we find for the projections of their homogeneous coordinate vectors

$$\langle \pi_q(x), \pi_q(y) \rangle = -\frac{\langle x, q\rangle \, \langle y, q\rangle}{\langle q, q\rangle}, \quad \langle \pi_q(x), \pi_q(x) \rangle = -\frac{\Delta_q(x)}{\langle q, q\rangle}, \quad \langle \pi_q(y), \pi_q(y) \rangle = -\frac{\langle y, q\rangle^2}{\langle q, q\rangle},$$

and thus

$$K_{\widetilde{\mathcal{Q}}}(\pi_{\boldsymbol{q}}(\boldsymbol{x}), \pi_{\boldsymbol{q}}(\boldsymbol{y})) = \frac{\langle \pi_q(x), \pi_q(y)\rangle^2}{\langle \pi_q(x), \pi_q(x)\rangle \, \langle \pi_q(y), \pi_q(y)\rangle} = \frac{\langle x, q\rangle^2}{\Delta_q(x)} = \mu.$$

Therefore, $\pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x})$ is a Cayley-Klein sphere with center $\pi_{\boldsymbol{q}}(\boldsymbol{x})$ and radius $\mu$.

We know that $\pi_{\boldsymbol{q}}(\boldsymbol{y}) \in \pi_{\boldsymbol{q}}(\mathcal{Q})$. Hence, according to Proposition 2.2, the sign of $\mu$, which is equal to the sign of $\Delta_q(x)$, determines which side of $\widetilde{\mathcal{Q}}$ the center $\pi_{\boldsymbol{q}}(\boldsymbol{x})$ lies on. Further we find that,

$$\mu = 0 \;\Leftrightarrow\; \boldsymbol{x} \in \boldsymbol{q}^\perp, \text{ and } \mu = 1 \;\Leftrightarrow\; \boldsymbol{x} \in \mathcal{Q},$$

which, again according to Proposition 2.2, corresponds to a hyperplane and the cone of contact respectively. $\square$

The map $\pi_{\boldsymbol{q}}^{\mathfrak{S}}$ covers the whole space of Cayley-Klein spheres with points in $\pi_{\boldsymbol{q}}(\mathcal{Q})$.

**Proposition 3.5.** *The map $\pi_{\boldsymbol{q}}^{\mathfrak{S}}$ constitutes a double cover of the set of Cayley-Klein spheres and horospheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ with respect to $\widetilde{\mathcal{Q}}$. Its ramification points are given by $(\boldsymbol{q}^\perp \cup \{\boldsymbol{q}\}) \cap \mathfrak{S}$, and its covering involution is $\sigma_{\boldsymbol{q}}$.*

*Proof.* We show that every Cayley-Klein sphere with points in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ possesses exactly two preimages, which are interchanged by $\sigma_{\boldsymbol{q}}$, unless it is a hyperplane. The same is true for Cayley-Klein horospheres.

Consider a Cayley-Klein sphere $S_\mu(\widetilde{\boldsymbol{x}})$ with center $\widetilde{\boldsymbol{x}} \in \boldsymbol{q}^\perp \setminus \widetilde{\mathcal{Q}}$, Cayley-Klein radius $\mu \in \mathbb{R}$ and points in $\pi_{\boldsymbol{q}}(\mathcal{Q})$. Then, according to Proposition 3.4, a preimage $\boldsymbol{x} \in \mathfrak{S}$, $\pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x}) = S_\mu(\widetilde{\boldsymbol{x}})$ must satisfy $\pi_{\boldsymbol{q}}(\boldsymbol{x}) = \widetilde{\boldsymbol{x}}$, i.e.

$$x = \widetilde{x} + \lambda q$$

for some $\lambda \in \mathbb{R}$, and

$$\frac{\langle x, q\rangle^2}{\Delta_q(x)} = \mu,$$

which is equivalent to

$$\lambda^2 = -\mu \frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle}.$$

According to Lemma 3.6 we have $-\mu \frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle} \geq 0$ since $S_\mu(\widetilde{\boldsymbol{x}}) \subset \pi_{\boldsymbol{q}}(\mathcal{Q})$, and thus

$$x_\pm \coloneqq \widetilde{x} \pm \sqrt{-\mu \frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle}}\, q$$

defines one or two (real) points $\boldsymbol{x}_\pm$ provided that $\boldsymbol{x}_\pm \in \mathfrak{S}$.

The two points are interchanged by the involution, $\sigma_{\boldsymbol{q}}(\boldsymbol{x}_\pm) = \boldsymbol{x}_\mp$, and we have

$$\boldsymbol{x}_+ = \boldsymbol{x}_- \;\Leftrightarrow\; \mu = 0,$$

and in this case $\boldsymbol{x}_\pm = \widetilde{\boldsymbol{x}} \in \boldsymbol{q}^\perp$.

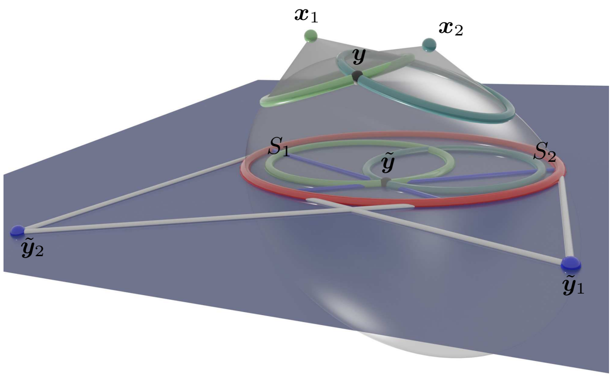



**Figure 7.** The Cayley-Klein distance with respect to $\mathcal{Q}$ corresponds to the Cayley-Klein intersection angle in the central projection to $\pi_{\boldsymbol{q}}(\mathcal{Q})$ (see Proposition 3.7).

To see that $\boldsymbol{x}_\pm^\perp \cap \mathcal{Q} \neq \varnothing$, first assume $\mu \neq 0$. We show that any point $\widetilde{\boldsymbol{y}} \in S_\mu(\widetilde{\boldsymbol{x}})$ on the Cayley-Klein sphere, has (real) preimages $\boldsymbol{y}_\pm \in \mathcal{Q}$, i.e. $\pi_{\boldsymbol{q}}(\boldsymbol{y}_\pm) = \widetilde{\boldsymbol{y}}$, that lie in the polar hyperplane of $\boldsymbol{x}_\pm$ respectively. Indeed, the points

$$y_\pm := \pm \langle q, q\rangle \sqrt{-\mu\frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle}}\widetilde{y} - \langle \widetilde{x}, \widetilde{y}\rangle q$$

satisfy

$$\langle y_\pm, y_\pm\rangle = -\langle q, q\rangle \left(\mu \langle \widetilde{x}, \widetilde{x}\rangle \langle \widetilde{y}, \widetilde{y}\rangle - \langle \widetilde{x}, \widetilde{y}\rangle^2\right) = 0,$$

and

$$\langle x_\pm, y_\pm\rangle = \pm \langle q, q\rangle \langle \widetilde{x}, \widetilde{y}\rangle \sqrt{-\mu\frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle}} \mp \langle q, q\rangle \langle \widetilde{x}, \widetilde{y}\rangle \sqrt{-\mu\frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle}} = 0.$$

If $\mu = 0$, then $\widetilde{\boldsymbol{x}} = \boldsymbol{x}_+ = \boldsymbol{x}_-$, and the whole line $\widetilde{\boldsymbol{y}} \wedge \boldsymbol{q}$ lies in the polar hyperplane of $\widetilde{\boldsymbol{x}}$. Since $\boldsymbol{y} \in \pi_{\boldsymbol{q}}(\mathcal{Q})$ the line $\widetilde{\boldsymbol{y}} \wedge \boldsymbol{q}$ has two real intersection points with $\mathcal{Q}$, which serve as preimages for $\widetilde{\boldsymbol{y}}$. $\square$

**Lemma 3.6.** *A Cayley-Klein sphere with center $\widetilde{\boldsymbol{x}} \in \boldsymbol{q}^\perp \setminus \widetilde{\mathcal{Q}}$ and Cayley-Klein radius $\mu \in \mathbb{R}$ has points in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ if and only if*

$$-\mu\frac{\langle \widetilde{x}, \widetilde{x}\rangle}{\langle q, q\rangle} \geq 0.$$

*Proof.* Follows from Proposition 2.2 and Proposition 3.2. $\square$

Thus, we have found that the lift of the Cayley-Klein space $\pi_{\boldsymbol{q}}(\mathcal{Q})$ to the quadric $\mathcal{Q}$ leads to a linearization of the corresponding Cayley-Klein spheres, in the sense that they become planar sections of $\mathcal{Q}$, which we represent by their polar points.

For two intersecting Cayley-Klein spheres we call the Cayley-Klein distance of the poles of the two tangent hyperplanes (with respect to the absolute quadric) their *Cayley-Klein intersection angle*. It is independent of the chosen intersection point. The Cayley-Klein distance of two points in $\mathfrak{S}$ describes exactly this Cayley-Klein intersection angle in the projection to $\pi_{\boldsymbol{q}}(\mathcal{Q})$ (see Figure 7),

**Proposition 3.7.** *Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{S}$ such that the corresponding $\mathcal{Q}$-spheres intersect. Let*

$$\boldsymbol{y} \in \mathcal{Q} \cap \boldsymbol{x}_1^\perp \cap \boldsymbol{x}_2^\perp$$

*be a point in that intersection, and $\tilde{\boldsymbol{y}} \coloneqq \pi_{\boldsymbol{q}}(\boldsymbol{y})$ its projection. Let $S_1, S_2$ be the two projected Cayley-Klein spheres corresponding to $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ respectively*

$$S_1 \coloneqq \pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x}_1), \qquad S_2 \coloneqq \pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x}_2).$$

*Let $\tilde{\boldsymbol{y}}_1$, $\tilde{\boldsymbol{y}}_2$ be the two poles of the tangent hyperplanes of $S_1$, $S_2$ at $\tilde{\boldsymbol{y}}$ respectively. Then*

$$K_{\mathcal{Q}}\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right) = K_{\widetilde{\mathcal{Q}}}\left(\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2\right).$$

*Proof.* First, we express the Cayley-Klein distance $K_{\mathcal{Q}}\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right)$ in terms of the projected centers $\tilde{\boldsymbol{x}}_1 \coloneqq \pi_{\boldsymbol{q}}(\boldsymbol{x}_1)$, $\tilde{\boldsymbol{x}}_2 \coloneqq \pi_{\boldsymbol{q}}(\boldsymbol{x}_2)$ and the projected intersection point $\tilde{\boldsymbol{y}}$. To this end, we write

$$x_1 = \tilde{x}_1 + \alpha_1 q, \qquad x_2 = \tilde{x}_2 + \alpha_2 q, \qquad y = \tilde{y} + \lambda q,$$

for some $\alpha_1, \alpha_2, \lambda \in \mathbb{R}$. From $\langle y, y\rangle = \langle x_1, y\rangle = \langle x_2, y\rangle = 0$ we obtain

$$\lambda^2 = -\frac{\langle \tilde{y}, \tilde{y}\rangle}{\langle q, q\rangle}, \qquad \alpha_1\lambda = -\frac{\langle \tilde{y}, \tilde{x}_1\rangle}{\langle q, q\rangle}, \qquad \alpha_2\lambda = -\frac{\langle \tilde{y}, \tilde{x}_2\rangle}{\langle q, q\rangle},$$

and therefore

$$\alpha_1\alpha_2 = -\frac{\langle \tilde{y}, \tilde{x}_1\rangle\ \langle \tilde{y}, \tilde{x}_2\rangle}{\langle \tilde{y}, \tilde{y}\rangle\ \langle q, q\rangle}, \qquad (\alpha_1)^2 = -\frac{\langle \tilde{y}, \tilde{x}_1\rangle^2}{\langle \tilde{y}, \tilde{y}\rangle\ \langle q, q\rangle}, \qquad (\alpha_2)^2 = -\frac{\langle \tilde{y}, \tilde{x}_2\rangle^2}{\langle \tilde{y}, \tilde{y}\rangle\ \langle q, q\rangle}.$$

Using this we find

$$\langle x_1, x_2\rangle = \langle \tilde{x}_1, \tilde{x}_2\rangle - \frac{\langle \tilde{y}, \tilde{x}_1\rangle\ \langle \tilde{y}, \tilde{x}_2\rangle}{\langle \tilde{y}, \tilde{y}\rangle}, \quad \langle x_1, x_1\rangle = \langle \tilde{x}_1, \tilde{x}_1\rangle - \frac{\langle \tilde{y}, \tilde{x}_1\rangle^2}{\langle \tilde{y}, \tilde{y}\rangle}, \quad \langle x_1, x_1\rangle = \langle \tilde{x}_1, \tilde{x}_1\rangle - \frac{\langle \tilde{y}, \tilde{x}_1\rangle^2}{\langle \tilde{y}, \tilde{y}\rangle},$$

and thus

$$K_{\mathcal{Q}}\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right) = \frac{\langle x_1, x_2\rangle^2}{\langle x_1, x_1\rangle\ \langle x_2, x_2\rangle} = \frac{\left(\langle \tilde{x}_1, \tilde{x}_2\rangle\ \langle \tilde{y}, \tilde{y}\rangle - \langle \tilde{y}, \tilde{x}_1\rangle\ \langle \tilde{y}, \tilde{x}_2\rangle\right)^2}{\left(\langle \tilde{x}_1, \tilde{x}_1\rangle\ \langle \tilde{y}, \tilde{y}\rangle - \langle \tilde{y}, \tilde{x}_1\rangle^2\right)\left(\langle \tilde{x}_2, \tilde{x}_2\rangle\ \langle \tilde{y}, \tilde{y}\rangle - \langle \tilde{y}, \tilde{x}_2\rangle^2\right)}. \tag{7}$$

Secondly, we express the right hand side $K_{\widetilde{\mathcal{Q}}}\left(\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2\right)$ in terms of the same quantities. From (4) we know that the poles $\tilde{\boldsymbol{y}}_1$, $\tilde{\boldsymbol{y}}_2$ of the tangent planes (with respect to $\widetilde{\mathcal{Q}}$) are given by

$$\tilde{y}_1 = \langle \tilde{x}_1, \tilde{y}\rangle\, \tilde{x}_1 - \mu_1 \langle \tilde{x}_1, \tilde{x}_1\rangle\, \tilde{y}, \qquad \tilde{y}_2 = \langle \tilde{x}_2, \tilde{y}\rangle\, \tilde{x}_2 - \mu_2 \langle \tilde{x}_2, \tilde{x}_2\rangle\, \tilde{y},$$

where

$$\mu_1 = \frac{\langle \tilde{x}_1, \tilde{y}\rangle^2}{\langle \tilde{x}_1, \tilde{x}_1\rangle\ \langle \tilde{y}, \tilde{y}\rangle}, \qquad \mu_2 = \frac{\langle \tilde{x}_2, \tilde{y}\rangle^2}{\langle \tilde{x}_2, \tilde{x}_2\rangle\ \langle \tilde{y}, \tilde{y}\rangle}$$

are the Cayley-Klein radii of $S_1$ and $S_2$. From this we obtain

$$\langle \tilde{y}_1, \tilde{y}_2\rangle = \langle \tilde{x}_1, \tilde{y}\rangle\ \langle \tilde{x}_2, \tilde{y}\rangle \left(\langle \tilde{x}_1, \tilde{x}_2\rangle - \frac{\langle \tilde{x}_1, \tilde{y}\rangle\ \langle \tilde{x}_2, \tilde{y}\rangle}{\langle \tilde{y}, \tilde{y}\rangle}\right),$$

$$\langle \tilde{y}_1, \tilde{y}_1\rangle = \langle \tilde{x}_1, \tilde{y}\rangle^2 \left(\langle \tilde{x}_1, \tilde{x}_1\rangle - \frac{\langle \tilde{x}_1, \tilde{y}\rangle^2}{\langle \tilde{y}, \tilde{y}\rangle}\right), \qquad \langle \tilde{y}_2, \tilde{y}_2\rangle = \langle \tilde{x}_2, \tilde{y}\rangle^2 \left(\langle \tilde{x}_2, \tilde{x}_2\rangle - \frac{\langle \tilde{x}_2, \tilde{y}\rangle^2}{\langle \tilde{y}, \tilde{y}\rangle}\right)$$

Substituting into

$$K_{\widetilde{\mathcal{Q}}}\left(\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2\right) = \frac{\langle \tilde{y}_1, \tilde{y}_2\rangle^2}{\langle \tilde{y}_1, \tilde{y}_1\rangle\ \langle \tilde{y}_2, \tilde{y}_2\rangle}$$

leads to the same as in (7). $\square$

*Remark* 3.5. Starting with two intersecting Cayley-Klein spheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ the lifted $\mathcal{Q}$-spheres must be chosen such that they intersect as well. Only then will the Cayley-Klein distance of the poles of the lifted spheres recover the Cayley-Klein intersection angle.

*Remark* 3.6. Every quadric comes with a naturally induced (pseudo-)conformal structure, see e.g. [Por1995]. The Cayley-Klein distance between the two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{S}$ also coincides with the angle measured in this conformal structure.

As a corollary of Theorem 1.3 we can now characterize the (local) transformations of a Cayley-Klein space $\pi_{\boldsymbol{q}}(\mathcal{Q})$ that map hyperspheres to hyperspheres as the projective orthogonal transformations in the lift to the quadric $\mathcal{Q}$.

**Theorem 3.8.** *Let $n \geq 2$, $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^{n+1}$ be a non-degenerate quadric, and $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+1} \setminus \mathcal{Q}$. Consider the Cayley-Klein space $\pi_{\boldsymbol{q}}(\mathcal{Q})$ endowed with the Cayley-Klein metric induced by $\widetilde{\mathcal{Q}} = \mathcal{Q} \cap \boldsymbol{q}^{\perp}$. Let $W \subset \pi_{\boldsymbol{q}}(\mathcal{Q})$ be an open subset, and $f : W \to \pi_{\boldsymbol{q}}(\mathcal{Q})$ be an injective map that maps intersections of Cayley-Klein hyperspheres with $W$ to intersections of Cayley-Klein hyperspheres with $f(W)$. Then $f$ is the restriction of a projective transformation $\mathbb{R}\mathrm{P}^{n+1} \to \mathbb{R}\mathrm{P}^{n+1}$ that preserves the quadric $\mathcal{Q}$.*

*Proof.* After lifting the open sets $W$ and $f(W)$ to $\mathcal{Q}$ the statement follows from Theorem 1.3. □

*Remark* 3.7. If the transformation $f$ is defined on the whole space $\pi_{\boldsymbol{q}}(\mathcal{Q})$ its lift must fix the point $\boldsymbol{q}$. Thus, in this case $f$ must be an isometry of $\pi_{\boldsymbol{q}}(\mathcal{Q})$.

*Remark* 3.8. If $n \geq 3$ the condition on $f$ of mapping hyperspheres to hyperspheres may be weakened to $f$ being a conformal transformation, i.e. preserving Cayley-Klein angles between arbitrary hypersurfaces (generalized Liouville's theorem, see [Por1995, Ben1992]).

*Remark* 3.9. The group of projective transformations $\mathrm{PO}(r,s)$ that preserve the quadric $\mathcal{Q}$ maps $\mathcal{Q}$-spheres to $\mathcal{Q}$-spheres. In the projection to $\pi_{\boldsymbol{q}}(\mathcal{Q})$ it may be interpreted as the group of transformations that map "oriented" points of $\pi_{\boldsymbol{q}}(\mathcal{Q})$ to "oriented" points of $\pi_{\boldsymbol{q}}(\mathcal{Q})$, while preserving Cayley-Klein spheres. It contains the subgroup $\mathrm{PO}(r,s)_{\boldsymbol{q}}$ of isometries of $\pi_{\boldsymbol{q}}(\mathcal{Q})$. The involution $\sigma_{\boldsymbol{q}}$ plays the role of "orientation reversion". For "hyperbolic Möbius geometry" see, e.g. [Som1914], and for "oriented" points of the hyperbolic plane [Yag1968].

## 3.3 Scaling along concentric spheres

The transformation group $\mathrm{PO}(r,s)$ contains the isometries of $\pi_{\boldsymbol{q}}(\mathcal{Q})$, given by $\mathrm{PO}(r,s)_{\boldsymbol{q}}$. It turns out that the only transformations additionally needed to generate the whole group $\mathrm{PO}(r,s)$ are "scalings" along concentric spheres.

In the lift to $\mathfrak{S}$ pencils of concentric Cayley-Klein spheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ correspond to lines in $\mathfrak{S}$ through $\boldsymbol{q}$ (cf. Proposition 3.4).

**Proposition 3.9.** *The preimage under the map $\pi_{\boldsymbol{q}}^{\mathfrak{S}}$ of a family of concentric Cayley-Klein spheres/horospheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ with center $\widetilde{\boldsymbol{x}} \in \boldsymbol{q}^{\perp}$ is given by the line $\ell \coloneqq (\widetilde{\boldsymbol{x}} \wedge \boldsymbol{q}) \cap \mathfrak{S}$.*

*For every $\boldsymbol{x} \in \ell$ the hyperplane $\boldsymbol{x}^{\perp}$ that defines the $\mathcal{Q}$-sphere by intersection with $\mathcal{Q}$ contains the polar subspace $(\widetilde{\boldsymbol{x}} \wedge \boldsymbol{q})^{\perp}$.*

**Definition 3.3.** We call a line in $\mathfrak{S}$ a *pencil of $\mathcal{Q}$-spheres*, and a line in $\mathfrak{S}$ containing the point $\boldsymbol{q}$ a *pencil of concentric $\mathcal{Q}$-spheres (with respect to $\boldsymbol{q}$)*.

For every pencil of $\mathcal{Q}$-spheres there is a distinguished one-parameter family of projective orthogonal transformations that preserve the pencil and each hyperplane through the corresponding line (see Figure 8).

**Proposition 3.10.** *Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{S}$ with $K_{\mathcal{Q}}(\boldsymbol{x}_1, \boldsymbol{x}_2) > 0$. Then there is a unique transformation $T_{\boldsymbol{x}_1,\boldsymbol{x}_2} \in \mathrm{PO}(r,s)$ that maps $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ and preserves every hyperplane through the line $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2$.*

*It satisfies $(T_{\boldsymbol{x}_1,\boldsymbol{x}_2})^{-1} = T_{\boldsymbol{x}_2,\boldsymbol{x}_1}$.*

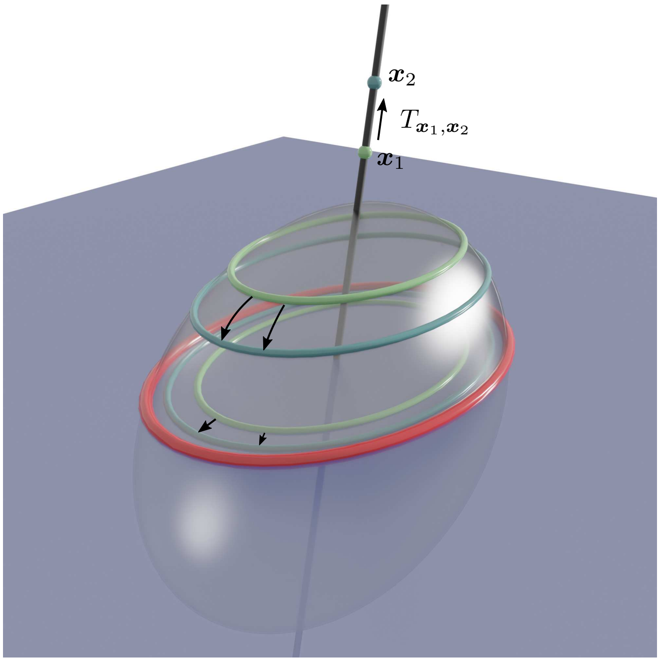



**Figure 8.** Scaling along a pencil of concentric Cayley-Klein spheres in the lift and in the projection.

**Definition 3.4.** For two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{S}$ on a pencil of concentric $\mathcal{Q}$-spheres with respect to $\boldsymbol{q}$, i.e. $\boldsymbol{q} \in \boldsymbol{x}_1 \wedge \boldsymbol{x}_2$, we call the transformation $T_{\boldsymbol{x}_1,\boldsymbol{x}_2} \in \mathrm{PO}(r,s)$ a *scaling along the pencil of concentric spheres* $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2$.

*Remark* 3.10. There are possibly three types of scalings along concentric spheres depending on the signature of the line $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2$.

In the projection $\pi_{\boldsymbol{q}}^{\mathfrak{S}}$ to $\pi_{\boldsymbol{q}}(\mathcal{Q})$ the line $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2$ through $\boldsymbol{q}$ corresponds to a pencil of concentric Cayley-Klein spheres. The transformation $T_{\boldsymbol{x}_1,\boldsymbol{x}_2}$ maps spheres of this pencil to spheres of this pencil.

Every transformation from $\mathrm{PO}(r,s)$ may be decomposed into a (lift of an) isometry of $\pi_{\boldsymbol{q}}(\mathcal{Q})$ and a scaling along a pencil of concentric spheres.

**Proposition 3.11.** *Let* $f \in \mathrm{PO}(r,s)$*. Then* $f$ *can be written as*

$$f = T_{\boldsymbol{q},\boldsymbol{x}} \circ \Phi = \Psi \circ T_{\boldsymbol{y},\boldsymbol{q}}$$

*with* $\boldsymbol{x} := f(\boldsymbol{q})$*,* $\boldsymbol{y} := f^{-1}(\boldsymbol{q})$ *and some* $\Phi, \Psi \in \mathrm{PO}(r,s)_{\boldsymbol{q}}$*.*

*Proof.* We observe that $T_{\boldsymbol{x},\boldsymbol{q}} \circ f, f \circ T_{\boldsymbol{q},\boldsymbol{y}} \in \mathrm{PO}(r,s)_{\boldsymbol{q}}$. □

*Remark* 3.11. To generate all transformations of $\mathrm{PO}(r,s)$ one may further restrict to (at most) three arbitrarily chosen one-parameter families of scalings (one of each type, cf. Remark 3.10). Then a transformation $f \in \mathrm{PO}(r,s)$ can be written as

$$f = \Phi \circ T \circ \Psi$$

where $\Phi, \Psi \in \mathrm{PO}(r,s)_{\boldsymbol{q}}$ and $T$ is exactly one of the three chosen scalings.

### 3.4 Möbius geometry

Let $\langle\cdot,\cdot\rangle$ be the standard non-degenerate bilinear form of signature $(n+1,1)$, i.e.

$$\langle x, y\rangle_{n,1} := x_1 y_1 + \ldots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}$$

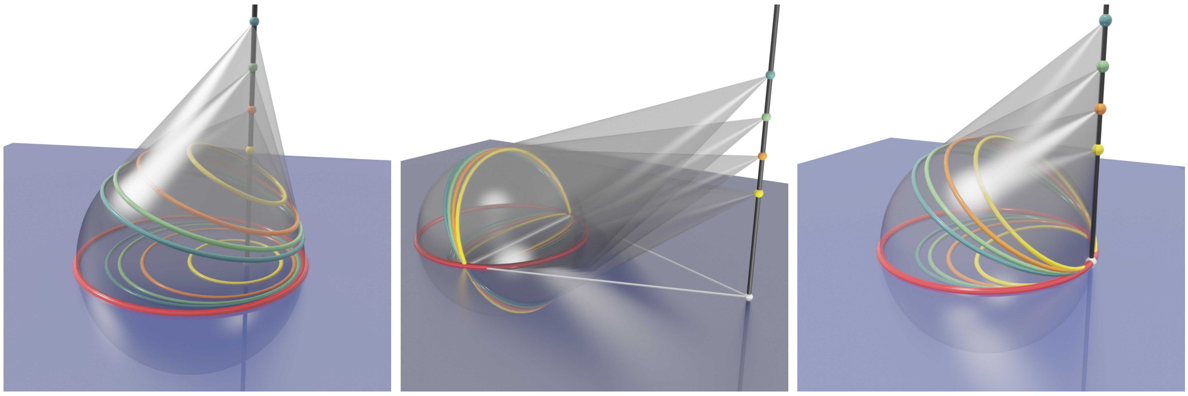


**Figure 9.** Hyperbolic geometry and its lift to Möbius geometry. *L*eft: Concentric hyperbolic circles. *M*iddle: Constant distance curves to a common line. *R*ight: Concentric horocycles with center on the absolute conic.

for $x, y \in \mathbb{R}^{n+2}$, and denote by $\mathcal{S} \subset \mathbb{R}\mathrm{P}^{n+1}$ the corresponding quadric, which we call the *Möbius quadric*.

The Möbius quadric is projectively equivalent to the standard round sphere $\mathcal{S} \simeq \mathbb{S}^n \subset \mathbb{R}^{n+1}$ (cf. Example 1.1). In this correspondence intersections of $\mathcal{S}$ with hyperplanes of $\mathbb{R}\mathrm{P}^{n+1}$, i.e. the $\mathcal{S}$-spheres, are identified with hyperspheres of $\mathbb{S}^n$ (cf. Proposition 5.2). The corresponding transformation group

$$\mathbf{Mob} := \mathrm{PO}(n+1,1)$$

of *Möbius transformations* leaves the quadric $\mathcal{S}$ invariant and maps hyperplanes to hyperplanes. Thus Möbius geometry may be seen as the geometry of points on $\mathbb{S}^n$ in which hyperspheres are mapped to hyperspheres.

The set of poles of hyperplanes that have non-empty intersections with the Möbius quadric $\mathcal{S}$ is given by the "outside" $\mathcal{S}^+$, Thus we identify the space of $\mathcal{S}$-spheres with

$$\mathfrak{S} = \mathcal{S}^+.$$

*Remark* 3.12. The Cayley-Klein metric on $\mathfrak{S}$ that is induced by the Möbius quadric $\mathcal{S}$ is called the *inversive distance*, see [Cox1971]. For two intersecting hyperspheres of $\mathbb{S}^n$ it is equal to the cosine of their intersection angle. For a signed version of this quantity see Section B.1. Comparing with Section 2.4 this same Cayley-Klein metric also induces $(n+1)$-dimensional hyperbolic geometry on the "inside" $\mathcal{S}^-$ of the Möbius quadric, and $(n+1)$-dimensional deSitter geometry on the "outside" $\mathfrak{S} = \mathcal{S}^+$ of the Möbius quadric.

Central projection of the $(n+1)$-dimensional Möbius quadric from a point leads to a double cover of $n$-dimensional hyperbolic/elliptic space.

### 3.5 Hyperbolic geometry and Möbius geometry

Given the Möbius quadric $\mathcal{S} \subset \mathbb{R}\mathrm{P}^n$ choose a point $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+1}$, $\langle q, q\rangle > 0$, w.l.o.g.

$$\boldsymbol{q} := [e_{n+1}] = [0, \dots 0, 1, 0].$$

The corresponding involution and projection take the form

$$\sigma_{\boldsymbol{q}} : [x_1, \dots, x_n, x_{n+1}, x_{n+2}] \mapsto [x_1, \dots, x_n, -x_{n+1}, x_{n+2}],$$
$$\pi_{\boldsymbol{q}} : [x_1, \dots, x_n, x_{n+1}, x_{n+2}] \mapsto [x_1, \dots, x_n, 0, x_{n+2}].$$

The quadric in the polar hyperplane of $\boldsymbol{q}$

$$\widetilde{\mathcal{S}} = \mathcal{S} \cap \boldsymbol{q}^\perp$$

has signature $(n, 1)$. Its "inside" $\mathcal{H} = \widetilde{\mathcal{S}}^-$ can be identified with $n$-dimensional hyperbolic space (cf. Section 2.4), and the Möbius quadric projects down to the compactified hyperbolic space

$$\overline{\mathcal{H}} = \pi_{\boldsymbol{q}}(\mathcal{S}).$$

According to Proposition 3.4, an $\mathcal{S}$-sphere, which we identify with a point in

$$\mathfrak{S} = \mathcal{S}^+ \cup \mathcal{S},$$

projects to the different types of *generalized hyperbolic spheres* in $\overline{\mathcal{H}}$ (see Figure 9 and Table 1).

**Proposition 3.12.** *Under the map*

$$\pi_{\boldsymbol{q}}^{\mathfrak{S}} : \boldsymbol{x} \mapsto \pi_{\boldsymbol{q}}(x^\perp \cap \mathcal{S})$$

*a point* $\boldsymbol{x} \in \mathfrak{S} = \mathcal{S}^+ \cup \mathcal{S}$

- *with* $\boldsymbol{x} \in \mathcal{S}$ *is mapped to a* ***point*** $\pi_{\boldsymbol{q}}(\boldsymbol{x}) \in \overline{\mathcal{H}}$*,*
- *with* $\boldsymbol{x} \in \boldsymbol{q}^\perp$*, i.e.* $x_{n+1} = 0$*, is mapped to a* ***hyperbolic hyperplane*** *in* $\overline{\mathcal{H}}$ *with pole* $\boldsymbol{x}$*,*
- *with* $\langle \pi_q(x), \pi_q(x) \rangle < 0$ *is mapped to a* ***hyperbolic sphere*** *in* $\overline{\mathcal{H}}$ *with center* $\pi_{\boldsymbol{q}}(\boldsymbol{x})$*. In the normalization* $\langle \pi_q(x), \pi_q(x) \rangle = -1$ *its hyperbolic radius is given by* $r \geq 0$*, where* $\cosh^2 r = x_{n+1}^2$*,*
- *with* $\langle \pi_q(x), \pi_q(x) \rangle > 0$ *is mapped to a* ***hyperbolic surface of constant distance*** *in* $\overline{\mathcal{H}}$ *to a hyperbolic hyperplane with pole* $\pi_{\boldsymbol{q}}(x)$*, In the normalization* $\langle \pi_q(x), \pi_q(x) \rangle = 1$ *its hyperbolic distance is given by* $r \geq 0$*, where* $\sinh^2 r = x_{n+1}^2$*.*
- *with* $\langle \pi_q(x), \pi_q(x) \rangle = 0$ *is mapped to a* ***hyperbolic horosphere****.*

*Proof.* Compare Section 2.4 for the different types of possible Cayley-Klein spheres in hyperbolic space. Following Proposition 3.4 they can be distinguished by the sign of the quadratic form $\Delta_q(x)$, or, comparing with equation (6), by the sign of $\langle \pi_q(x), \pi_q(x) \rangle$. Furthermore, the center of the Cayley-Klein sphere corresponding to $\boldsymbol{x}$ is given by $\pi_{\boldsymbol{q}}(\boldsymbol{x})$, while its Cayley-Klein radius is given by

$$\mu = \frac{\langle x, q \rangle^2}{\Delta_q(x)} = -\frac{\langle x, q \rangle^2}{\langle q, q \rangle \; \langle \pi_q(x), \pi_q(x) \rangle} = -\frac{x_{n+1}^2}{\langle \pi_q(x), \pi_q(x) \rangle}.$$

□

*Remark* 3.13.

(i) The map $\pi_{\boldsymbol{q}}^{\mathfrak{S}}$ is a double cover of the set of generalized hyperbolic spheres, branching on the subset of hyperbolic planes (see Proposition 3.5).

(ii) The Cayley-Klein distance induced on $\mathfrak{S}$ by $\mathcal{S}$ measures the Cayley-Klein angle between the corresponding generalized hyperbolic spheres (see Proposition 3.7) if their lifts intersect (see Remark 3.5), and more generally their inversive distance (see Remark 3.12).

(iii) In the projection to $\overline{\mathcal{H}}$ Möbius transformations map generalized hyperbolic spheres to generalized hyperbolic spheres (see Remark 3.9). Vice versa, every (local) transformation of the hyperbolic space that maps generalized hyperbolic spheres to generalized hyperbolic spheres is the restriction of a Möbius transformation (see Theorem 3.8)

(iv) Every Möbius transformation can be decomposed into two hyperbolic isometries and a scaling along either a fixed pencil of concentric hyperbolic spheres, distance surfaces, or horospheres (see Remark 3.11).

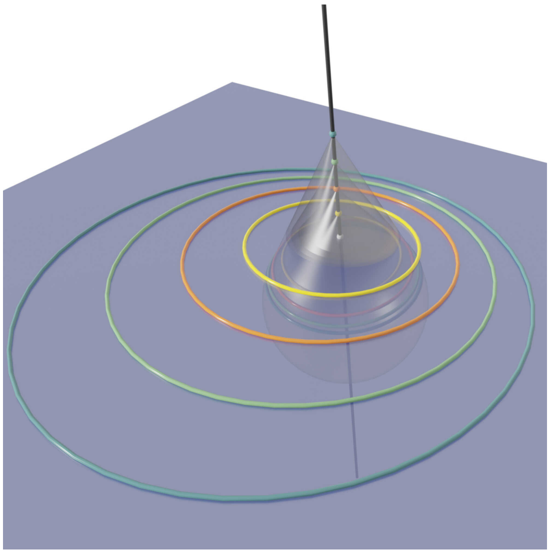


**Figure 10.** Concentric elliptic circles in elliptic geometry and its lift to Möbius geometry.

| **hyperbolic geometry** | **Möbius geometry** |
|---|---|
| *point*<br>$\boldsymbol{y} \in \mathcal{H}$, $y = (\hat{y}, y_{n+1}) \in \mathbb{H}^n$ | $[\hat{y}, \pm 1, y_{n+1}] \in \mathcal{S}$ |
| *hyperplane*<br>with pole $\boldsymbol{y} \in \mathrm{dS}$, $y = (\hat{y}, y_{n+1}) \in \mathrm{d}\mathbb{S}^n$ | $[\hat{y}, 0, y_{n+1}] \in \mathcal{S}^+ \cap \boldsymbol{q}^\perp$ |
| *sphere*<br>with center $\boldsymbol{y} \in \mathcal{H}$, $y = (\hat{y}, y_{n+1}) \in \mathbb{H}^n$<br>and radius $r > 0$ | $[\hat{y}, \pm \cosh r, y_{n+1}] \in \mathcal{S}^+ \cap C_{\mathcal{S}}(\boldsymbol{q})^+$ |
| *surface of constant distance*<br>$r > 0$ to a hyperplane<br>with pole $\boldsymbol{y} \in \mathrm{dS}$, $y = (\hat{y}, y_{n+1}) \in \mathrm{d}\mathbb{S}^n$ | $[\hat{y}, \pm \sinh r, y_{n+1}] \in \mathcal{S}^+ \cap C_{\mathcal{S}}(\boldsymbol{q})^-$ |
| *horosphere*<br>with center $\boldsymbol{y} \in \widetilde{\mathcal{S}}$, $y = (\hat{y}, y_{n+1}) \in \mathbb{L}^{n,1}$ | $[\hat{y}, \pm e^r, y_{n+1}] \in \mathcal{S}^+ \cap C_{\mathcal{S}}(\boldsymbol{q})$ |

**Table 1.** The lifts of generalized hyperbolic spheres to Möbius geometry.

### 3.6 Elliptic geometry and Möbius geometry

Given the Möbius quadric $\mathcal{S} \subset \mathbb{R}\mathrm{P}^n$ choose a point $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+1}$, $\langle q, q \rangle < 0$, w.l.o.g.

$$\boldsymbol{q} := [e_{n+1}] = [0, \dots 0, 0, 1].$$

The corresponding involution and projection take the form

$$\begin{aligned} \sigma_{\boldsymbol{q}} &: [x_1, \dots, x_{n+1}, x_{n+2}] \mapsto [x_1, \dots, x_{n+1}, -x_{n+2}] \\ \pi_{\boldsymbol{q}} &: [x_1, \dots, x_{n+1}, x_{n+2}] \mapsto [x_1, \dots, x_{n+1}, 0] \end{aligned}$$

The quadric in the polar hyperplane of $\boldsymbol{q}$

$$\widetilde{\mathcal{S}} = \mathcal{S} \cap \boldsymbol{q}^\perp$$

is imaginary, and has signature $(n+1, 0)$. The Möbius quadric projections down to its "outside"

$$\mathcal{E} = \widetilde{\mathcal{S}}^+,$$

which can be identified with $n$-dimensional elliptic space (cf. Section 2.5).

According to Proposition 3.4 an $\mathcal{S}$-sphere projects to an elliptic sphere in $\mathcal{E}$ (see Figure 10 and Table 2).

**Proposition 3.13.** *Under the map*

$$\pi_{\boldsymbol{q}}^{\mathfrak{S}} : x \mapsto \pi_{\boldsymbol{q}}(x^{\perp} \cap \mathcal{S})$$

*a point $\boldsymbol{x} \in \mathfrak{S} = \mathcal{S}^{+} \cup \mathcal{S}$*

- *with $\boldsymbol{x} \in \mathcal{S}$ is mapped to a **point** $\pi_{\boldsymbol{q}}(\boldsymbol{x}) \in \mathcal{E}$,*
- *with $\boldsymbol{x} \in \boldsymbol{q}^{\perp}$, i.e. $x_{n+2} = 0$, is mapped to an **elliptic plane** in $\mathcal{E}$ with pole $\boldsymbol{x}$,*
- *with $\langle \pi_q(x), \pi_q(x) \rangle > 0$ is mapped to an **elliptic sphere** in $\mathcal{E}$ with center $\pi_{\boldsymbol{q}}(\boldsymbol{x})$, In the normalization $\langle \pi_q(x), \pi_q(x) \rangle = 1$ its elliptic radius is given by $r \geq 0$, where $\cos^2 r = x_{n+2}^2$. It also has constant elliptic distance $R \geq 0$, where $\sin^2 R = x_{n+1}^2$, to the polar hyperplane of $\pi_{\boldsymbol{q}}(\boldsymbol{x})$.*

*Remark* 3.14.

(i) The map $\pi_{\boldsymbol{q}}^{\mathfrak{S}}$ is a double cover of the set of elliptic spheres, branching on the subset of elliptic planes (see Proposition 3.5).

(ii) The Cayley-Klein distance induced on $\mathfrak{S}$ by $\mathcal{S}$ measures the Cayley-Klein angle between the corresponding elliptic spheres (see Proposition 3.7) if their lifts intersect (see Remark 3.5), and more generally their inversive distance (see Remark 3.12).

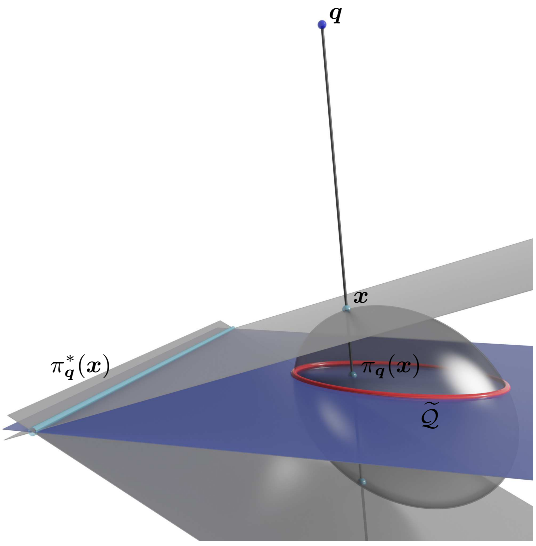
(iii) In the projection to $\mathcal{E}$, Möbius transformations map elliptic spheres to elliptic spheres (see Remark 3.9) Vice versa, every (local) transformation of elliptic space that maps elliptic spheres to elliptic spheres is the restriction of a Möbius transformation (see Theorem 3.8).

(iv) Every Möbius transformation can be decomposed into two elliptic isometries and a scaling along one fixed pencil of concentric elliptic spheres (see Remark 3.11).

*Remark* 3.15. Upon the identification of the Möbius quadric with the sphere $\mathcal{S} \simeq \mathbb{S}^n$ the group of Möbius transformations fixing the point $\boldsymbol{q}$ is the group of spherical transformations, yielding spherical geometry, which is a double cover of elliptic geometry.

| **elliptic geometry** | **Möbius geometry** |
|---|---|
| *point*<br>$\boldsymbol{y} \in \mathcal{E}$, $y \in \mathbb{S}^n$ | $[y, \pm 1] \in \mathcal{S}$ |
| *hyperplane*<br>with pole $\boldsymbol{y} \in \mathcal{E}$, $y \in \mathbb{S}^n$ | $[y, 0] \in \boldsymbol{q}^{\perp}$ |
| *sphere*<br>with center $\boldsymbol{y} \in \mathcal{E}$, $y \in \mathbb{S}^n$<br>and radius $r > 0$ | $[y, \pm \cos r] \in \mathcal{S}^{+}$ |

**Table 2.** The lifts of elliptic spheres to Möbius geometry.



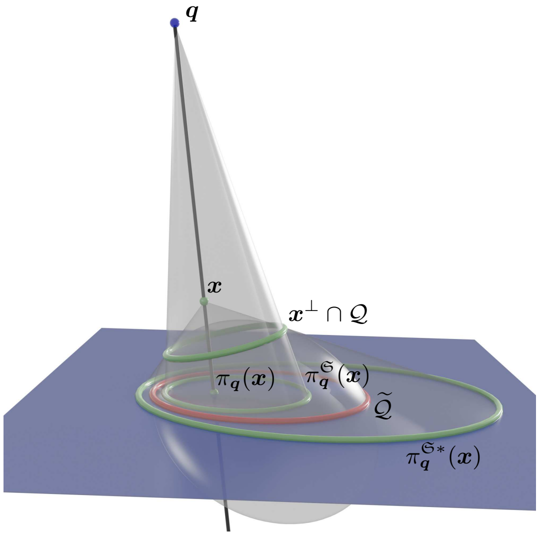



**Figure 11.** *L*eft: Polar projection of points on the quadric $\mathcal{Q}$. *R*ight: Polar projection of $\mathcal{Q}$-spheres.

# 4 Laguerre geometry in space forms

The primary objects in Möbius geometry are points on $\mathcal{S}$, which are a double cover of the points in hyperbolic/elliptic space, and spheres, which are correspondingly a double cover of the spheres in hyperbolic/elliptic space. The primary incidence between these objects is *a point lying on a sphere*.

Laguerre geometry is dual to Möbius geometry in the sense that the primary objects are hyperplanes, and spheres, both being a double cover of the corresponding objects in hyperbolic/elliptic space, while the primary incidence between these objects is *a plane being tangent to a sphere*.

While the double cover of points in a space form in the case of Möbius geometry may be interpreted as “oriented points” (cf. Remark 3.9), in the case of Laguerre geometry this leads to the perhaps more intuitive notion of “oriented hyperplanes”.

## 4.1 Polar projection

Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^{n+1}$ be a quadric. We have seen that the projection $\pi_{\boldsymbol{q}}$ of the quadric $\mathcal{Q}$ leads to a double cover of the points of $\pi_{\boldsymbol{q}}(\mathcal{Q}) \subset \boldsymbol{q}^\perp$ (cf. Proposition 3.1), i.e. the points “inside” or “outside” the quadric

$$\widetilde{\mathcal{Q}} = \mathcal{Q} \cap \boldsymbol{q}^\perp.$$

Correspondingly, the $\mathcal{Q}$-spheres yield a double cover of the Cayley-Klein spheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ (cf. Proposition 3.5). We now investigate the corresponding properties for polar hyperplanes and polar Cayley-Klein spheres.

**Definition 4.1.** Let $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^{n+1}$ be a quadric and $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+1} \setminus \mathcal{Q}$. Then we call the map

$$\pi_{\boldsymbol{q}}^* : \boldsymbol{x} \mapsto \boldsymbol{x}^\perp \cap \boldsymbol{q}^\perp = (\boldsymbol{x} \wedge \boldsymbol{q})^\perp,$$

that maps a point $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^{n+1}$ to the intersection of its polar hyperplane $\boldsymbol{x}^\perp$ with $\boldsymbol{q}^\perp$, the *polar projection (associated with the point $\boldsymbol{q}$)*.

The projection $\pi_{\boldsymbol{q}}$ and the polar projection $\pi_{\boldsymbol{q}}^*$ map the same point to a point in $\boldsymbol{q}^\perp$ and its polar hyperplane respectively.

**Proposition 4.1.** *For a point $\boldsymbol{x} \in \mathbb{R}\mathrm{P}^{n+1}$ its projection $\pi_{\boldsymbol{q}}(\boldsymbol{x}) \in \boldsymbol{q}^\perp$ is the pole of its polar projection $\pi_{\boldsymbol{q}}^*(\boldsymbol{x}) \subset \boldsymbol{q}^\perp$, where polarity in $\boldsymbol{q}^\perp \simeq \mathbb{R}\mathrm{P}^n$ is taken with respect to $\widetilde{\mathcal{Q}}$.*

If we restrict the polar projection $\pi_{\boldsymbol{q}}^*$ to the quadric $\mathcal{Q}$ we obtain a map to the hyperplanes of $\boldsymbol{q}^\perp$, which are poles of image points of the projection $\pi_{\boldsymbol{q}}$ (see Figure 11). This map leads to a double cover of the polar hyperplanes (cf. Proposition 3.1).

**Proposition 4.2.** *The restriction of the polar projection onto the quadric $\pi_{\boldsymbol{q}}^*\big|_{\mathcal{Q}}$ is a double cover of the set of all hyperplanes that are polar to the points in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ with branch locus $\mathcal{Q} \cap \boldsymbol{q}^\perp$.*

*Remark* 4.1. The double cover can be interpreted as carrying the additional information of the orientation of these hyperplanes, where the involution $\sigma_{\boldsymbol{q}}$ plays the role of *orientation reversion.*

By polarity every point $\boldsymbol{x} \in \mathfrak{S}$ corresponds to a $\mathcal{Q}$-sphere $\boldsymbol{x}^\perp \cap \mathcal{Q}$ (see Definition 3.2). In the projection to $\pi_{\boldsymbol{q}}(\mathcal{Q})$ it becomes a Cayley-Klein sphere (see Proposition 3.4), which is obtained from the point $\boldsymbol{x}$ by the map

$$\pi_{\boldsymbol{q}}^{\mathfrak{S}} : \boldsymbol{x} \mapsto \pi_{\boldsymbol{q}}(\boldsymbol{x}^\perp \cap \mathcal{Q})$$

The polar projection $\pi_{\boldsymbol{q}}^*$ of each point of a $\mathcal{Q}$-sphere yields a tangent plane of the polar Cayley-Klein sphere of $\pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x})$ (see Definition 2.4). The points of the polar Cayley-Klein sphere are therefore obtained by the map

$$\pi_{\boldsymbol{q}}^{\mathfrak{S}*} : \boldsymbol{x} \mapsto C_{\mathcal{Q}}(\boldsymbol{x}),$$

where $C_{\mathcal{Q}}(\boldsymbol{x})$ is the cone of contact (see Definition 1.1) to $\mathcal{Q}$ with vertex $\boldsymbol{x}$ (see Figure 11).

**Proposition 4.3.** *For $\boldsymbol{x} \in \mathfrak{S}$ the two Cayley-Klein spheres/horospheres $\pi_{\boldsymbol{q}}^{\mathfrak{S}}(\boldsymbol{x})$ and $\pi_{\boldsymbol{q}}^{\mathfrak{S}*}(\boldsymbol{x})$ are mutually polar Cayley-Klein spheres in $\boldsymbol{q}^\perp$ with respect to $\widetilde{\mathcal{Q}}$.*

This leads to a polar version of Proposition 3.5.

**Proposition 4.4.** *The map $\pi_{\boldsymbol{q}}^{\mathfrak{S}*}$ constitutes a double cover of the set of Cayley-Klein spheres/horospheres which are polar to Cayley-Klein spheres/horospheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ with respect to $\widetilde{\mathcal{Q}}$. Its ramification points are given by $(\boldsymbol{q}^\perp \cup \{\boldsymbol{q}\}) \cap \mathfrak{S}$, and its covering involution is $\sigma_{\boldsymbol{q}}$.*

*Remark* 4.2. Following Remark 4.1 we may endow the Cayley-Klein spheres/horospheres that are polar to Cayley-Klein spheres/horospheres in $\pi_{\boldsymbol{q}}(\mathcal{Q})$ with an orientation by lifting them to planar sections of $\mathcal{Q}$, i.e. $\mathcal{Q}$-spheres. We call the planar section, or equivalently their oriented projections *Laguerre spheres (of $\pi_{\boldsymbol{q}}(\mathcal{Q})$).* The involution $\sigma_{\boldsymbol{q}}$ acts on Laguerre spheres as orientation reversion.

The Cayley-Klein distance of two points in $\mathfrak{S}$ describes the Cayley-Klein tangent distance between the two corresponding Cayley-Klein spheres in the projection to $\pi_{\boldsymbol{q}}(\mathcal{Q})$. This is the polar version of Proposition 3.7.

**Proposition 4.5.** *Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathfrak{S}$ such that the corresponding $\mathcal{Q}$-spheres intersect. Let*

$$\boldsymbol{y} \in \mathcal{Q} \cap \boldsymbol{x}_1 \cap \boldsymbol{x}_2$$

*be a point in that intersection, and $\tilde{\boldsymbol{y}} \coloneqq \pi_{\boldsymbol{q}}^*(\boldsymbol{y})$ its polar projection. Let $S_1, S_2$ be the two polar projected Cayley-Klein spheres corresponding to $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ respectively*

$$S_1 \coloneqq \pi_{\boldsymbol{q}}^{\mathfrak{S}*}(\boldsymbol{x}_1), \qquad S_2 \coloneqq \pi_{\boldsymbol{q}}^{\mathfrak{S}*}(\boldsymbol{x}_2).$$

*Let $\tilde{\boldsymbol{y}}_1$, $\tilde{\boldsymbol{y}}_2$ be the two tangent points of $\tilde{\boldsymbol{y}}$ to $S_1$, $S_2$ respectively. Then*

$$K_{\mathcal{Q}}\left(\boldsymbol{x}_1, \boldsymbol{x}_2\right) = K_{\widetilde{\mathcal{Q}}}\left(\tilde{\boldsymbol{y}}_1, \tilde{\boldsymbol{y}}_2\right).$$

*Proof.* Consider Proposition 3.7. By polarity, the intersection point of the spheres becomes a common tangent hyperplane, and the intersection angle becomes the distance of the two tangent points. □

*Remark* 4.3. Following Remark 4.1 and Remark 4.2 a common point in the lift of two (oriented) Laguerre spheres corresponds to a common oriented tangent hyperplane. Thus the Cayley-Klein distance on $\mathfrak{S}$ measures the Cayley-Klein tangent distance between two (oriented) Laguerre spheres (cf. Remark 3.5).

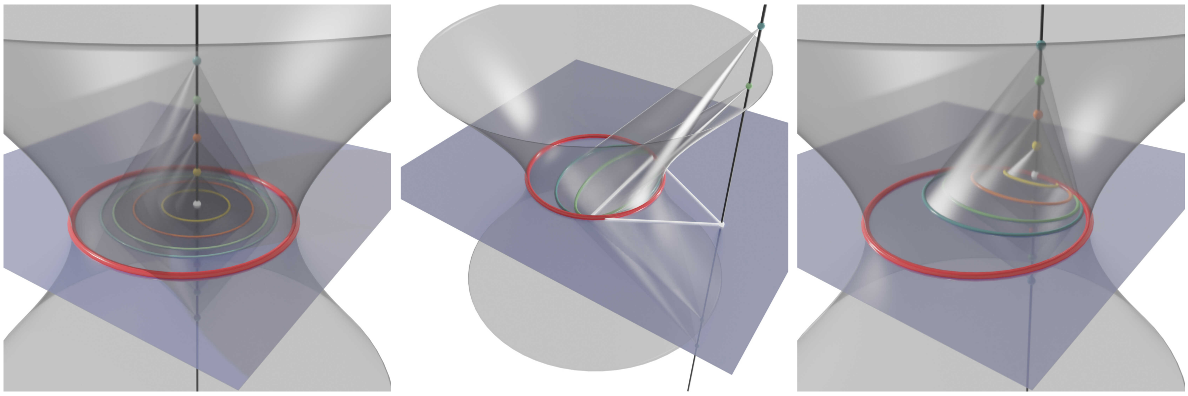


**Figure 12.** Hyperbolic Laguerre geometry. *L*eft: Concentric hyperbolic circles. *M*iddle: Constant distance curves to a common line. *R*ight: Concentric horocycles with center on the absolute conic.

### 4.2 Hyperbolic Laguerre geometry

When projecting down from Möbius geometry to hyperbolic geometry (cf. Section 3.5) we obtain a double cover of the points in hyperbolic space. Hyperbolic planes, on the other hand, are represented by points in deSitter space, or "outside" hyperbolic space, by polarity. Thus, to obtain hyperbolic Laguerre geometry, instead of the Möbius quadric, we choose a quadric that projects to deSitter space from some point.

**Definition 4.2.**

(i) We call the quadric

$$\mathcal{B}_{\mathrm{hyp}} \subset \mathbb{R}\mathrm{P}^{n+1}$$

corresponding to the standard bilinear form of signature $(n, 2)$ in $\mathbb{R}^{n+2}$, i.e.,

$$\langle x, y \rangle := x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1} - x_{n+2} y_{n+2}$$

for $x, y \in \mathbb{R}^{n+2}$, the *hyperbolic Laguerre quadric.*

(ii) The corresponding transformation group

$$\mathbf{Lag}_{\mathrm{hyp}} := \mathrm{PO}(n, 2)$$

is called the group of *hyperbolic Laguerre transformations.*

To recover hyperbolic space in the projection, choose a point $\boldsymbol{p} \in \mathbb{R}\mathrm{P}^{n+1}$ with $\langle p, p \rangle < 1$, w.l.o.g.,

$$\boldsymbol{p} := [e_{n+2}] = [0, \ldots, 0, 1].$$

The corresponding involution and projection take the form

$$\begin{aligned}
\sigma_{\boldsymbol{p}} &: [x_1, \ldots, x_{n+1}, x_{n+2}] \mapsto [x_1, \ldots, , x_{n+1}, -x_{n+2}],\\
\pi_{\boldsymbol{p}} &: [x_1, \ldots, x_{n+1}, x_{n+2}] \mapsto [x_1, \ldots, , x_{n+1}, 0].
\end{aligned}$$

The quadric in the polar hyperplane of $\boldsymbol{p}$

$$\widetilde{\mathcal{S}} := \mathcal{B}_{\mathrm{hyp}} \cap \boldsymbol{p}^{\perp}$$

has signature $(n, 1)$, and its "inside" $\mathcal{H} = \widetilde{\mathcal{S}}^{-}$ can be identified with $n$-dimensional hyperbolic space, while its "outside" $\mathrm{dS} = \widetilde{\mathcal{S}}^{+}$ can be identified with $n$-dimensional deSitter space (cf. Section 2.4).

Under the projection $\pi_{\boldsymbol{p}}$ the hyperbolic Laguerre quadric projects down to the compactified deSitter space

$$\pi_{\boldsymbol{p}}(\mathcal{B}_{\text{hyp}}) = \overline{\text{dS}} = \text{dS} \cup \widetilde{\mathcal{S}}.$$

Thus the polar projection $\pi_{\boldsymbol{p}}^*$ of a point on $\mathcal{B}_{\text{hyp}}$ yields a hyperbolic hyperplane, where the double cover can be interpreted as encoding the orientation of that hyperplane.

*Remark* 4.4. The quadric $\mathcal{B}_{\text{hyp}}$ is the projective version of the hyperboloid $\widetilde{\text{d}\mathbb{S}}^n$ introduced in Section 2.4 as a double cover of deSitter space.

We call the hyperplanar sections of $\mathcal{B}_{\text{hyp}}$, i.e. the $\mathcal{B}_{\text{hyp}}$-spheres, *hyperbolic Laguerre spheres.* By polarity we identify the space of hyperbolic Laguerre spheres with the whole space

$$\mathfrak{S} = \mathbb{R}\text{P}^{n+1}. \tag{8}$$

Under the polar projection $\pi_{\boldsymbol{p}}^{\mathfrak{S}*}$ points in $\mathfrak{S}$ are mapped to the spheres which are polar to deSitter spheres (see Figure 12 and Table 3).

**Proposition 4.6.** *Under the map*

$$\pi_{\boldsymbol{p}}^{\mathfrak{S}*} : \boldsymbol{x} \mapsto C_{\mathcal{B}_{\text{hyp}}}(\boldsymbol{x})$$

*a point $\boldsymbol{x} \in \mathfrak{S} = \mathbb{R}\text{P}^{n+1}$*

- ► *with $\boldsymbol{x} \in \mathcal{B}_{\text{hyp}}$ is mapped to a **hyperbolic hyperplane** with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x}) \in \overline{\text{dS}}$,*
- ► *with $\boldsymbol{x} \in \boldsymbol{p}^{\perp}$, i.e. $x_{n+2} = 0$, is mapped to a deSitter **null-sphere** in $\overline{\text{dS}}$ with center $\boldsymbol{x}$,*
- ► *with $\langle \pi_p(x), \pi_p(x) \rangle < 0$ is mapped to a **hyperbolic sphere** in $\overline{\mathcal{H}}$ with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$. In the normalization $\langle \pi_p(x), \pi_p(x) \rangle = -1$ its hyperbolic radius is given by $r \geq 0$, where $\sinh^2 r = x_{n+2}^2$,*
- ► *with $\langle \pi_p(x), \pi_p(x) \rangle > 0$ and $\langle x, x \rangle < 0$ is mapped to a **hyperbolic surface of constant distance** in $\overline{\mathcal{H}}$ to a hyperbolic hyperplane with pole $\pi_{\boldsymbol{p}}(x)$, In the normalization $\langle \pi_p(x), \pi_p(x) \rangle = 1$ its hyperbolic distance is given by $r \geq 0$, where $\cosh^2 r = x_{n+2}^2$.*
- ► *with $\langle \pi_p(x), \pi_p(x) \rangle > 0$ and $\langle x, x \rangle > 0$ is mapped to a **deSitter sphere** in $\overline{\text{dS}}$ with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$. In the normalization $\langle \pi_p(x), \pi_p(x) \rangle = 1$ its deSitter radius is given by $r \geq 0$, where $\cos^2 r = x_{n+2}^2$.*
- ► *with $\langle \pi_p(x), \pi_p(x) \rangle = 0$ is mapped to a **hyperbolic horosphere**.*

*Remark* 4.5.

(i) The points $\boldsymbol{x}$ representing hyperbolic spheres/distance hypersurfaces/horospheres can be distinguished from the points representing deSitter sphere by the first satisfying $\langle x, x \rangle < 0$, i.e. lying "inside" the hyperbolic Laguerre quadric, and the latter satisfying $\langle x, x \rangle > 0$, i.e. lying "outside" the hyperbolic Laguerre quadric.

(ii) The map $\pi_{\boldsymbol{p}}^{\mathfrak{S}*}$ is a double cover of the spheres described in Proposition 4.6, branching on the subset of hyperbolic points, and deSitter null-spheres (see Proposition 4.4). We interpret the lift as being endowed with an orientation and call the corresponding oriented spheres *hyperbolic Laguerre spheres.* The involution $\sigma_{\boldsymbol{p}}$ acts on the set of hyperbolic Laguerre spheres as orientation reversion. Upon the normalization given in Proposition 4.6 the sign of the $x_{n+2}$-component can be interpreted as the orientation of the corresponding hyperbolic Laguerre sphere.

(iii) The Cayley-Klein distance induced on $\mathfrak{S}$ by the hyperbolic Laguerre quadric $\mathcal{B}_{\text{hyp}}$ measures the Cayley-Klein tangent distance between the corresponding hyperbolic Laguerre spheres (see Proposition 4.5) if they possess a common oriented tangent hyperplane (see Remark 4.3).

(iv) The hyperbolic Laguerre quadric $\mathcal{B}_{\text{hyp}}$ contains one-dimensional isotropic subspaces (cf. Lemma 1.2). Those are mapped under the polar projection $\pi^*_{\boldsymbol{p}}$ to parallel hyperbolic subspaces (see Figure 12).

(v) Using the projection $\pi_{\boldsymbol{p}}$ instead of the polar projection $\pi^*_{\boldsymbol{p}}$ hyperbolic Laguerre geometry may also be interpreted as the "Möbius geometry" corresponding to deSitter space (cf. Section 5.3).

| **hyperbolic geometry** | **Laguerre geometry** |
|---|---|
| *hyperplane* with pole $\boldsymbol{y} \in \text{dS}$, $y \in \text{d}\mathbb{S}^n$ | $[y, \pm 1] \in \mathcal{B}_{\text{hyp}}$ |
| *sphere* with center $\boldsymbol{y} \in \mathcal{H}$, $y \in \mathbb{H}^n$ and radius $r > 0$ | $[y, \pm \sinh r] \in \mathcal{B}^-_{\text{hyp}} \cap C_{\mathcal{B}_{\text{hyp}}}(\boldsymbol{p})^-$ |
| *surface of constant distance* $r > 0$ to a hyperplane with pole $\boldsymbol{y} \in \text{dS}$, $y \in \text{d}\mathbb{S}^n$ | $[y, \pm \cosh r] \in \mathcal{B}^-_{\text{hyp}} \cap C_{\mathcal{B}_{\text{hyp}}}(\boldsymbol{p})^+$ |
| *horosphere* with center $\boldsymbol{y} \in \widetilde{\mathcal{S}}$ | $[y, \pm e^r] \in \mathcal{B}^-_{\text{hyp}} \cap C_{\mathcal{B}_{\text{hyp}}}(\boldsymbol{p})$ |
| *deSitter null-sphere* with center $\boldsymbol{y} \in \text{dS}$, $y \in \text{d}\mathbb{S}^n$ | $[y, 0] \in \mathcal{B}^+_{\text{hyp}} \cap \boldsymbol{p}^\perp$ |
| *deSitter sphere* with center $\boldsymbol{y} \in \text{dS}$, $y \in \text{d}\mathbb{S}^n$ and deSitter radius $r > 0$ | $[y, \pm \cos r] \in \mathcal{B}^+_{\text{hyp}}$ |

**Table 3.** The lifts of polar deSitter spheres to hyperbolic Laguerre geometry.

### 4.2.1 Hyperbolic Laguerre transformations

Every (local) transformation mapping (non-oriented) hyperbolic hyperplanes to hyperbolic hyperplanes (not necessarily points to points) while preserving (tangency to) hyperbolic spheres can be lifted and extended to a hyperbolic Laguerre transformation (see Theorem 3.8).

The hyperbolic Laguerre group

$$\mathbf{Lag}_{\text{hyp}} = \text{PO}(n, 2)$$

preserves the hyperbolic Laguerre quadric $\mathcal{B}_{\text{hyp}}$ and maps planar sections of $\mathcal{B}_{\text{hyp}}$ to planar section of $\mathcal{B}_{\text{hyp}}$. Under the polar projection this means it maps oriented hyperplanes to oriented hyperplanes, or (oriented) hyperbolic Laguerre spheres to hyperbolic Laguerre spheres, while preserving the tangent distance and in particular the oriented contact (see Remark 4.3).

The hyperbolic Laguerre group contains (doubly covers) the group of hyperbolic isometries as $\text{PO}(n, 2)_{\boldsymbol{p}}$. To generate the whole Laguerre group we only need to add three specific one-parameter families of scalings along concentric Laguerre spheres (see Remark 3.11 and Figure 12).

▶ Consider the family of transformations

$$T_t^{(\mathrm{s})} := \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_n & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & \cos t & \sin t \\ 0 & \cdots & 0 & -\sin t & \cos t \end{array}\right] \qquad \text{for } t \in [-\pi/2, \pi/2].$$

It maps the absolute $\boldsymbol{p} = [0, \dots, 0, 1]$ to

$$T_t^{(\mathrm{s})}(\boldsymbol{p}) = [0, \dots, 0, \sin t, \cos t],$$

which is a hyperbolic sphere with center $[0, \dots, 1, 0]$. It turns from the absolute for $t = 0$ into a point for $t = \pm\pi/2$, while changing orientation when it passes through the center or through the absolute, i.e. when $\cos t \cdot \sin t$ changes sign.

▶ Consider the family of transformations

$$T_t^{(\mathrm{c})} := \left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0 \\ & I_{n-1} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & \cosh t & 0 & \sinh t \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & \sinh t & 0 & \cosh t \end{array}\right] \qquad \text{for } t \in \mathbb{R}$$

It maps the absolute $\boldsymbol{p} = [0, \dots, 0, 1]$ to

$$T_t^{(\mathrm{c})}(\boldsymbol{p}) = [0, \dots, 0, \sinh t, 0, \cosh t],$$

which is an oriented hypersurface of constant distance to the hyperbolic hyperplane $[0, \dots, 0, 1, 0, 1]$. It turns from the absolute for $t = 0$ into the hyperplane for $t = \infty$, while changing orientation when it passes through the absolute, i.e. when $t$ changes sign.

▶ Consider the family of transformations

$$T_t^{(\mathrm{h})} := \left[\begin{array}{ccc|cc|c} & & & 0 & 0 & 0 \\ & I_{n-1} & & \vdots & \vdots & \vdots \\ & & & 0 & 0 & 0 \\ \hline 0 & \cdots & 0 & 1 + \frac{t^2}{2} & \frac{t^2}{2} & t \\ 0 & \cdots & 0 & -\frac{t^2}{2} & 1 - \frac{t^2}{2} & -t \\ \hline 0 & \cdots & 0 & t & t & 1 \end{array}\right] \qquad \text{for } t \in \mathbb{R}.$$

It maps the absolute $\boldsymbol{p} = [0, \dots, 0, 1]$ to

$$T_t^{(\mathrm{h})}(\boldsymbol{s}) = [0, \dots, 0, t, -t, 1],$$

which is a horosphere with center $[0, \dots, 0, 1, -1, 0]$ on the absolute. It turns from the absolute for $t = 0$ into the center for $t = \infty$, while changing orientation when $t$ changes its sign.

*Remark* 4.6. While Laguerre transformations preserve oriented contact they do not preserve the notion of sphere, horosphere and constant distance surface. For example the transformation $T_{\pi/2}^{(\mathrm{s})}$ transforms the origin into the absolute and thus turns all spheres which contain the origin into horospheres.

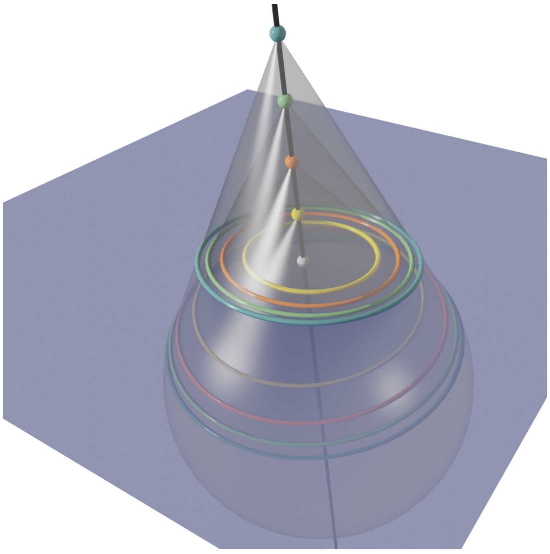


**Figure 13.** Concentric elliptic circles in elliptic Laguerre geometry.

Now the hyperbolic Laguerre group can be generated by hyperbolic motions and the three introduced one-parameter families of scalings (see Remark 3.11).

**Theorem 4.7.** *Every hyperbolic Laguerre transformation $f \in \mathrm{PO}(n,2)$ can be written as*

$$f = \Phi T_t \Psi,$$

*where $\Phi, \Psi \in \mathrm{PO}(n,2)_{\boldsymbol{p}}$ are hyperbolic motions and $T_t \in \{T_t^{(\mathrm{s})}, T_t^{(\mathrm{c})}, T_t^{(\mathrm{h})}\}$ a scaling for some $t \in \mathbb{R}$.*

## 4.3 Elliptic Laguerre geometry

When projecting down from Möbius geometry to elliptic geometry (cf. Section 3.6) we obtain a double cover of the points in elliptic space. Since every elliptic hyperplane has a pole in the elliptic space, this equivalently leads to a double cover of the elliptic hyperplanes.

**Definition 4.3.**

(i) We call the quadric

$$\mathcal{B}_{\mathrm{ell}} \subset \mathbb{R}\mathrm{P}^{n+1}$$

corresponding to the standard bilinear form of signature $(n+1,1)$ in $\mathbb{R}^{n+2}$, i.e.,

$$\langle x, y \rangle \coloneqq x_1 y_1 + \ldots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2}$$

for $x, y \in \mathbb{R}^{n+2}$ the *elliptic Laguerre quadric.*

(ii) The corresponding transformation group

$$\mathbf{Lag}_{\mathrm{ell}} \coloneqq \mathrm{PO}(n+1,1) \simeq \mathbf{Mob}$$

is called the group of *elliptic Laguerre transformations.*

*Remark* 4.7. Thus, $n$-dimensional elliptic Laguerre geometry is isomorphic to $n$-dimensional Möbius geometry.

To recover elliptic space in the projection, choose a point $\boldsymbol{p} \in \mathbb{R}\mathrm{P}^{n+1}$ with $\langle p, p \rangle < 1$, w.l.o.g.,

$$\boldsymbol{p} \coloneqq [e_{n+2}] = [0, \dots, 0, 1].$$

The corresponding involution and projection take the form

$$\sigma_{\boldsymbol{p}} : [x_1, \dots, x_{n+1}, x_{n+2}] \mapsto [x_1, \dots, , x_{n+1}, -x_{n+2}],$$
$$\pi_{\boldsymbol{p}} : [x_1, \dots, x_{n+1}, x_{n+2}] \mapsto [x_1, \dots, , x_{n_1}, 0].$$

The quadric in the polar hyperplane of $\boldsymbol{p}$

$$\mathcal{O} = \mathcal{B}_{\text{ell}} \cap \boldsymbol{p}^{\perp}$$

has signature $(n+1, 0)$, and its non-empty side $\mathcal{E} = \mathcal{O}^{+}$ can be identified with $n$-dimensional elliptic space (see Section 2.5)

Under the projection $\pi_{\boldsymbol{p}}$ the elliptic Laguerre quadric projects down to the elliptic space

$$\pi_{\boldsymbol{p}}(\mathcal{B}_{\text{ell}}) = \mathcal{E}$$

Thus the polar projection $\pi_{\boldsymbol{p}}^{*}$ of every point on $\mathcal{B}_{\text{ell}}$ yields an elliptic hyperplane, where the double cover can be interpreted as carrying the orientation of that hyperplane.

*Remark* 4.8. The quadric $\mathcal{B}_{\text{ell}}$ is the projective version of the sphere $\mathbb{S}^n$ introduced in Section 2.5 as a double cover of elliptic space.

We call the hyperplanar sections of $\mathcal{B}_{\text{ell}}$, i.e. the $\mathcal{B}_{\text{ell}}$-spheres, *elliptic Laguerre spheres.* By polarity we identify the space of hyperbolic Laguerre spheres with the the outside of $\mathcal{B}_{\text{ell}}$

$$\mathfrak{S} = \mathcal{B}_{\text{ell}}^{+} \cup \mathcal{B}_{\text{ell}}. \tag{9}$$

Under the polar projection $\pi_{\boldsymbol{p}}^{\mathfrak{S}*}$ points in $\mathfrak{S}$ are mapped to spheres that are polar to elliptic spheres, i.e. they are mapped to all elliptic spheres (see Figure 13 and Table 4).

**Proposition 4.8.** *Under the map*

$$\pi_{\boldsymbol{p}}^{\mathfrak{S}*} : \boldsymbol{x} \mapsto C_{\mathcal{B}_{\text{hyp}}}(\boldsymbol{x})$$

*a point $\boldsymbol{x} \in \mathfrak{S} = \mathbb{R}\mathrm{P}^{n+1}$ is mapped to an **elliptic sphere** in $\mathcal{E}$ with center $\pi_{\boldsymbol{p}}(\boldsymbol{x})$. In the normalization $\langle \pi_p(x), \pi_p(x) \rangle = 1$ its elliptic radius is given by $r \geq 0$, where $x_{n+2}^2 = \sin^2 r$. In particular,*

- *$\boldsymbol{x} \in \mathcal{B}_{\text{ell}}$ is mapped to an **elliptic hyperplane** with pole $\pi_{\boldsymbol{p}}(\boldsymbol{x}) \in \mathcal{E}$,*
- *$\boldsymbol{x} \in \boldsymbol{p}^{\perp} = \mathcal{E}$, i.e. $x_{n+2} = 0$, is mapped to an **elliptic point** $\boldsymbol{x} \in \mathcal{E}$.*

*Remark* 4.9.

(i) The map $\pi_{\boldsymbol{p}}^{\mathfrak{S}*}$ is a double cover of elliptic spheres, branching on the subset of elliptic points (see Proposition 4.4). We interpret the lift as being endowed with an orientation and call the corresponding oriented spheres *elliptic Laguerre spheres.* The involution $\sigma_{\boldsymbol{q}}$ acts on the set of elliptic Laguerre spheres as orientation reversion. Upon the normalization given in Proposition 4.8 the sign of the $x_{n+2}$-component can be interpreted as the orientation of the corresponding elliptic Laguerre sphere.

(ii) The Cayley-Klein distance induced on $\mathfrak{S}$ by the elliptic Laguerre quadric $\mathcal{B}_{\text{ell}}$ measures the Cayley-Klein tangent distance between the corresponding elliptic Laguerre spheres (see Proposition 4.5) if they possess a common oriented tangent hyperplane (see Remark 4.3).

(iii) Using the projection $\pi_{\boldsymbol{q}}$ instead of the polar projection $\pi_{\boldsymbol{q}}^*$ elliptic Laguerre geometry coincides with Möbius geometry (cf. Section 3.4 and Section 5.3).

| **elliptic geometry** | **Laguerre geometry** |
|---|---|
| *hyperplane* with pole $\boldsymbol{y} \in \mathcal{E}$, $y \in \mathbb{S}^n$ | $[y, \pm 1] \in \mathcal{B}_{\mathrm{ell}}$ |
| *sphere* with center $\boldsymbol{y} \in \mathcal{E}$, $y \in \mathbb{S}^n$ and radius $r > 0$ | $[y, \pm \sin r] \in \mathcal{B}_{\mathrm{ell}}^+$ |
| *point* $\boldsymbol{y} \in \mathcal{E}$, $y \in \mathbb{S}^n$ | $[y, 0] \in \mathcal{B}_{\mathrm{ell}}^+ \cap \boldsymbol{p}^\perp$ |

**Table 4.** The lifts of elliptic spheres to elliptic Laguerre geometry.

### 4.3.1 Elliptic Laguerre transformations

Every (local) transformation mapping (non-oriented) elliptic hyperplanes to elliptic hyperplanes (not necessarily points to points) while preserving (tangency to) elliptic spheres can be lifted and extended to an elliptic Laguerre transformation (see Theorem 3.8).

The elliptic Laguerre group

$$\mathbf{Lag}_{\mathrm{ell}} = \mathrm{PO}(n+1, 1)$$

preserves the elliptic Laguerre quadric $\mathcal{B}_{\mathrm{ell}}$ and maps planar sections of $\mathcal{B}_{\mathrm{ell}}$ to planar section of $\mathcal{B}_{\mathrm{ell}}$. Under the polar projection this means it maps oriented hyperplanes to oriented hyperplanes, or (oriented) elliptic Laguerre spheres to elliptic Laguerre spheres, while preserving the tangent distance and in particular the oriented contact (see Remark 4.3).

The elliptic Laguerre group contains (doubly covers) the group of elliptic isometries as $\mathrm{PO}(n+1, 1)_{\boldsymbol{p}}$. To generate the whole Laguerre group we only need to add one specific one-parameter family of scalings along concentric Laguerre spheres (see Remark 3.11 and Figure 13).

► Consider the family of transformations

$$S_t := \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_n & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & \cosh t & \sinh t \\ 0 & \cdots & 0 & \sinh t & \cosh t \end{array}\right] \qquad \text{for } t \in \mathbb{R}.$$

It maps the absolute $\boldsymbol{p} = [0, \dots, 0, 1]$ to

$$T_t^{(\mathrm{s})}(\boldsymbol{p}) = [0, \dots, 0, \sinh t, \cosh t],$$

which is an elliptic sphere with center $[0, \dots, 1, 0]$. It turns from the absolute for $t = 0$ into a point for $t = \infty$, while changing orientation when it passes through the center or through the absolute, i.e. when $t$ changes sign.

Now the elliptic Laguerre group can be generated by elliptic motions and this one-parameter family of scalings (see Remark 3.11).

**Theorem 4.9.** *Any elliptic Laguerre transformation $f \in \mathrm{PO}(n+1, 1)$ can be written as*

$$f = \Phi S_t \Psi,$$

*where $\Phi, \Psi \in \mathrm{PO}(n+1, 1)_{\boldsymbol{p}}$ are elliptic motions and $t \in \mathbb{R}$.*

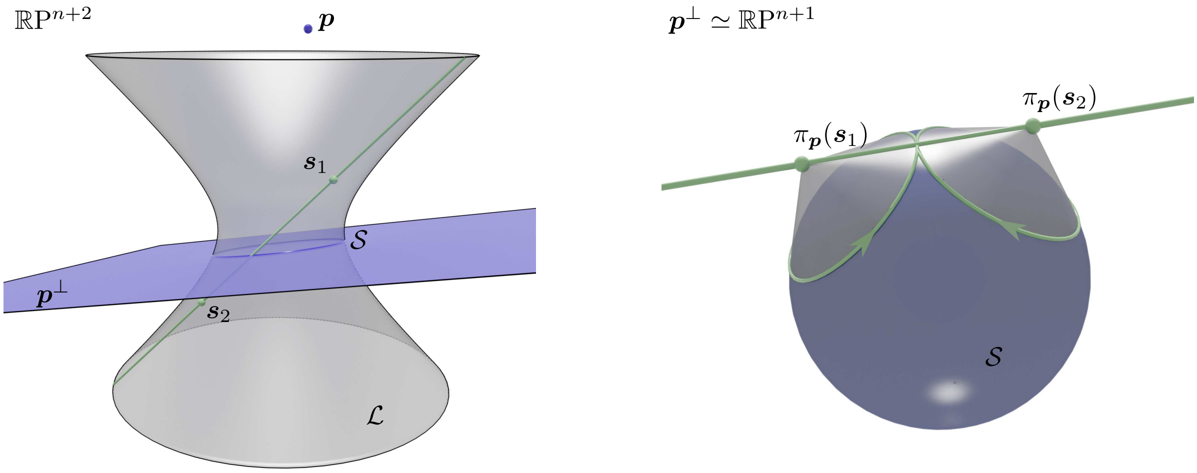



**Figure 14.** *Left*: The Lie quadric $\mathcal{L} \subset \mathbb{RP}^{n+2}$ (depicted in the case $n = 1$). The choice of $\boldsymbol{p}$ with $\langle p, p\rangle < 0$ determines the point complex, or Möbius quadric, $\mathcal{S} \subset \mathcal{L}$. The points $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{L}$ contained in a common isotropic subspace of $\mathcal{L}$ correspond to two oriented hyperspheres in oriented contact. *Right*: The Möbius quadric $\mathcal{S} \subset \mathbb{RP}^{n+1}$ (depicted in the case $n = 2$). The two points $\pi_{\boldsymbol{p}}(\boldsymbol{s}_1), \pi_{\boldsymbol{p}}(\boldsymbol{s}_2) \in \mathcal{S}^+$ correspond to two hyperspheres in $\mathcal{S}$ via polarity. In the chosen normalization, their orientation is encoded in the last component of $s_1$, $s_2$ respectively.

# 5 Lie geometry

Möbius geometry (signature $(n+1, 1)$, see Section 3.4), hyperbolic Laguerre geometry (signature $(n, 2)$, see Section 4.2), elliptic Laguerre geometry (signature $(n+1, 1)$, see Section 4.3), as well as Euclidean Laguerre geometry (signature $(n, 1, 1)$, see Section A.3) can all be lifted to Lie geometry (signature $(n+2, 2)$) in the sense of Sections 3 and 4.

We first give an intuitive description of Lie (sphere) geometry as the geometry of oriented hyperspheres of the $n$-sphere $\mathbb{S}^n$ and their oriented contact.

## 5.1 Oriented hyperspheres of $\mathbb{S}^n$

Consider the $n$-dimensional sphere

$$\mathbb{S}^n = \left\{ y \in \mathbb{R}^{n+1} \;\middle|\; y \cdot y = 1 \right\} \subset \mathbb{R}^{n+1},$$

where $y \cdot y$ denotes the standard scalar product on $\mathbb{R}^{n+1}$. An oriented hypersphere of $\mathbb{S}^n$ can be represented by its center $c \in \mathbb{S}^n$ and its signed spherical radius $r \in \mathbb{R}$ (see Figure 14). Tuples $(c, r) \in \mathbb{S}^n \times \mathbb{R}$ represent the same oriented hypersphere if they are related by a sequence of the transformations

$$\rho_1 : (c, r) \mapsto (c, r + 2\pi), \quad \rho_2 : (c, r) \mapsto (-c, r - \pi). \tag{10}$$

The corresponding hypersphere as a set of points is given by

$$\{ y \in \mathbb{S}^n \mid c \cdot y = \cos r \},$$

while its orientation is obtained in the following way: The hypersphere separates the sphere $\mathbb{S}^n$ into two regions. For $r \in [0, \pi)$ consider the region which contains the center $c$ to be the "inside" of the hypersphere, and endow the hypersphere with an orientation by assigning normals pointing towards the other region, the "outside" of the hypersphere. The orientation of the hypersphere for other values of $r$ is then obtained by (10).

**Definition 5.1.** We call
$$\vec{\mathscr{S}} := {(\mathbb{S}^n \times \mathbb{R})}\big/{\{\rho_1, \rho_2\}}.$$
the *space of oriented hyperspheres* of $\mathbb{S}^n$.

*Remark* 5.1. Orientation reversion defines an involution on $\vec{\mathscr{S}}$, which is given by
$$\rho : (c, r) \mapsto (c, -r).$$
Thus, the *space of (non-oriented) hyperspheres* of $\mathbb{S}^n$ may be represented by
$$\mathscr{S} := \vec{\mathscr{S}}\big/\rho = {(\mathbb{S}^n \times \mathbb{R})}\big/{\{\rho, \rho_1, \rho_2\}}.$$

Two oriented hyperspheres $(c_1, r_1)$ and $(c_2, r_2)$ are in *oriented contact* if (see Figure 14)
$$c_1 \cdot c_2 = \cos(r_1 - r_2), \tag{11}$$
which is a well-defined relation on $\vec{\mathscr{S}}$. Upon introducing coordinates $(c, \cos r, \sin r)$ on $\vec{\mathscr{S}}$ the transformations (10) may be replaced by
$$(c, \cos r, \sin r) \mapsto (-c, \cos(r - \pi), \sin(r - \pi)) = -(c, \cos r, \sin r), \tag{12}$$
while (11) becomes a bilinear relation, i.e.
$$c_1 \cdot c_2 - \cos r_1 \cos r_2 - \sin r_1 \sin r_2 = 0. \tag{13}$$
This gives rise to a projective model of Lie geometry as described in the following.

**Definition 5.2.**

(i) The quadric
$$\mathcal{L} \subset \mathbb{RP}^{n+2}$$
corresponding to the standard bilinear form of signature $(n + 1, 2)$
$$\langle x, y\rangle := \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2} - x_{n+3} y_{n+3}$$
for $x, y \in \mathbb{R}^{n+3}$, is called the *Lie quadric*.

(ii) Two points $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{L}$ on the Lie quadric are called *Lie orthogonal* if $\langle s_1, s_2\rangle = 0$, or equivalently if the line $\boldsymbol{s}_1 \wedge \boldsymbol{s}_2$ is isotropic, i.e. is contained in $\mathcal{L}$. An isotropic line is called a *contact element*.

(iii) The projective transformations of $\mathbb{RP}^{n+2}$ that preserve the Lie quadric $\mathcal{L}$
$$\mathbf{Lie} := \mathrm{PO}(n + 1, 2).$$
are called *Lie transformations*.

**Proposition 5.1.** *The set of oriented hyperspheres $\vec{\mathscr{S}}$ of $\mathbb{S}^n$ is in one-to-one correspondence with the Lie quadric $\mathcal{L}$ by the map*
$$\vec{S} : \vec{\mathscr{S}} \to \mathcal{L}, \quad (c, r) \mapsto (c, \cos r, \sin r)$$
*such that two oriented hyperspheres are in oriented contact if and only if their corresponding points on the Lie quadric are Lie orthogonal.*

*Proof.* A point $\boldsymbol{s} \in \mathcal{L}$ can always be represented by $\boldsymbol{s} = [c, \cos r, \sin r]$ with $c \in \mathbb{S}^n$, $r \in \mathbb{R}$. Now the statement follows from (12) and (13). □

| **spherical geometry** | **Lie geometry** |
|---|---|
| *point* $\hat{x} \in \mathbb{S}^n$ | $[\hat{x}, 1, 0] \in \mathcal{L}$ |
| *oriented hypersphere* with center $\hat{s} \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ | $[\hat{s}, \cos r, \sin r] \in \mathcal{L}$ |

**Table 5.** Correspondence of hyperspheres of the $n$-sphere $\mathbb{S}^n$ and points on the Lie quadric $\mathcal{L}$.

This correspondence leads to an embedding of $\mathbb{S}^n$ into the Lie quadric in the following way. Among all oriented hyperspheres the map $\vec{S}$ distinguishes the set of "points", or *null-spheres*, as the set of oriented hyperspheres with radius $r = 0$. It turns out that

$$\left\{\vec{S}(c, 0) \;\middle|\; c \in \mathbb{S}^n\right\} = \{\boldsymbol{x} \in \mathcal{L} \mid x_{n+3} = 0\} = \mathcal{L} \cap \boldsymbol{p}^\perp,$$

where

$$\boldsymbol{p} \coloneqq [e_{n+3}] = [0, \ldots, 0, 1] \in \mathbb{R}\mathrm{P}^{n+2}.$$

**Definition 5.3.** The quadric

$$\mathcal{S} \coloneqq \mathcal{L} \cap \boldsymbol{p}^\perp$$

is called the *point complex*.

*Remark* 5.2. Every choice of a timelike point $\boldsymbol{p} \in \mathbb{R}\mathrm{P}^{n+2}$, i.e. $\langle p, p\rangle < 0$, leads to the definition of a point complex $\mathcal{S} = \mathcal{L} \cap \boldsymbol{p}^\perp$, all of which are equivalent up to a Lie transformation. The chosen point complex $\mathcal{S}$ then leads to a correspondence of points on the Lie quadric $\mathcal{L}$ and oriented hyperspheres on $\mathcal{S} \simeq \mathbb{S}^n$.

The point complex is a quadric of signature $(n + 1, 1)$ which we identify with the Möbius quadric (see Section 3.4). The corresponding involution and projection associated with the point $\boldsymbol{p}$ (see Definition 3.1) take the form

$$\begin{aligned} \sigma_{\boldsymbol{p}} &: [x_1, \ldots, x_{n+2}, x_{n+3}] \mapsto [x_1, \ldots, , x_{n+2}, -x_{n+3}], \\ \pi_{\boldsymbol{p}} &: [x_1, \ldots, x_{n+2}, x_{n+3}] \mapsto [x_1, \ldots, , x_{n+2}, 0]. \end{aligned}$$

The image of the Lie quadric $\mathcal{L}$ under the projection $\pi_{\boldsymbol{p}}$ is given by

$$\pi_{\boldsymbol{p}}(\mathcal{L}) = \mathcal{S}^+ \cup \mathcal{S} = \left\{\boldsymbol{s} \in \boldsymbol{p}^\perp \;\middle|\; \langle s, s\rangle \geq 0\right\}.$$

By polarity, each point $\mathcal{S}^+ \cup \mathcal{S}$ corresponds to a hyperplanar section of $\mathcal{S}$. Thus, the Lie quadric can be seen as a double cover of the set of spheres of Möbius geometry, encoding their orientation, while, vice versa, the orientation of hyperspheres in Lie geometry vanishes in the projection to Möbius geometry.

**Proposition 5.2.**

*(i) The involution $\sigma_{\boldsymbol{p}} : \mathcal{L} \to \mathcal{L}$ corresponds to the orientation reversion on $\vec{\mathscr{S}}$.*

*(ii) The projection $\pi_{\boldsymbol{p}} : \mathcal{L} \to \mathcal{S}^+ \cup \mathcal{S}$ defines a double cover with branch locus $\mathcal{S}$.*

*(iii) The set $\mathscr{S}$ of non-oriented hyperspheres of $\mathbb{S}^n$ (see Remark 5.1) is in one-to-one correspondence with $\mathcal{S}^+ \cup \mathcal{S}$ by the map*

$$S = \pi_{\boldsymbol{p}} \circ \vec{S} : \mathscr{S} \to \mathcal{S}^+ \cup \mathcal{S}, \quad (c, r) \mapsto (c, \cos r, 0).$$

*(iv)* *The set of "points" on $\mathcal{S} \subset \mathcal{L}$ lying on an oriented hypersphere $\boldsymbol{s} \in \mathcal{L}$, or equivalently lying on the non-oriented hypersphere $\pi_{\boldsymbol{p}}(\boldsymbol{s}) \in \mathcal{S}^+ \cup \mathcal{S}$ is given by*

$$\boldsymbol{s}^\perp \cap \mathcal{S} = \pi_{\boldsymbol{p}}(\boldsymbol{s})^\perp \cap \mathcal{S}.$$

*(v)* *The non-oriented hyperspheres corresponding to two points $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}^+ \cup \mathcal{S}$ touch if and only if the line $\boldsymbol{s}_1 \wedge \boldsymbol{s}_2$ connecting them is tangent to $\mathcal{S}$.*

*Thus, the points on the cone of contact $C_{\mathcal{S}}(\boldsymbol{s})$ (see Definition 1.1) correspond to all spheres touching the sphere corresponding to $\boldsymbol{s} \in \mathcal{S}^+ \cup \mathcal{S}$.*

*Proof.*

(i) Note that $\sigma_{\boldsymbol{p}}(\vec{S}(c,r)) = \vec{S}(c,-r)$ and compare with Remark 5.1.

(ii) See Proposition 3.1 (ii).

(iii) Follows from (i) and (ii).

(iv) The set $\boldsymbol{s}^\perp \cap \mathcal{S} \subset \mathcal{L}$ describes all hyperspheres in oriented contact with $\boldsymbol{s}$ that simultaneously correspond to "points", i.e. "points" that lie on the hypersphere. Indeed, with $\boldsymbol{s} = [\widehat{s}, \cos r, \sin r] \in \mathcal{L}$ we find for a "point" $\boldsymbol{x} = [\widehat{x}, 1, 0] \in \mathcal{S}$ that

$$\langle s, x\rangle = 0 \;\Leftrightarrow\; \langle \widehat{s}, \widehat{x}\rangle = \cos r.$$

(v) This generalizes the statement in (iv) and follows from the fact that the isotropic subspaces of $\mathcal{L}$ (contact elements, cf. Definition 5.2) project to tangent lines of $\mathcal{S}$.

□

The subgroup $\mathbf{Lie}_{\boldsymbol{p}}$ of Lie transformations that preserve the point complex $\mathcal{S}$, i.e. map "points" to "points", becomes the group of Möbius transformations in the projection to $\boldsymbol{p}^\perp$

$$\mathbf{Mob} = {}^{\mathbf{Lie}_{\boldsymbol{p}}}\!/\!_{\sigma_{\boldsymbol{p}}} \simeq \mathrm{PO}(n+1,1).$$

## 5.2 Laguerre geometry from Lie geometry

A sphere complex in Lie geometry is given by the intersection of the Lie quadric with a hyperplane of $\mathbb{RP}^{n+2}$. It may equivalently be described by the polar point of this hyperplane. Two points in $\mathbb{RP}^{n+2}$ can be mapped to each other by a Lie transformation if and only if they have the same signature. Thus, any two sphere complexes of the same signature are Lie equivalent.

**Definition 5.4.** For a point $\boldsymbol{q} \in \mathbb{RP}^{n+2}$ the set of points

$$\mathcal{L} \cap \boldsymbol{q}^\perp$$

on the Lie quadric as well as the $n$-parameter family of oriented hyperspheres corresponding to these points is called a *sphere complex*. A sphere complex is further called

- *elliptic* if $\langle q, q\rangle > 0$,
- *hyperbolic* if $\langle q, q\rangle < 0$,
- *parabolic* if $\langle q, q\rangle = 0$.

*Remark* 5.3.

(i) We adopted the classical naming convention for sphere complexes here, see e.g. [Bla1929].

(ii) The point complex (see Definition 5.3) is a hyperbolic sphere complex.

(iii) A non-parabolic sphere complex induces an invariant for pairs of oriented spheres (see Section B).

Laguerre geometry is the geometry of oriented hyperplanes and oriented hyperspheres in a certain space form, and their oriented contact (cf. Section 4). It appears as a subgeometry of Lie geometry by distinguishing the set of "oriented hyperplanes" as a sphere complex among the set of oriented hyperspheres.

The point complex $\mathcal{S} = \mathcal{L} \cap \boldsymbol{p}^\perp$, where $\boldsymbol{p} \in \mathbb{R}\mathrm{P}^{n+2}$ is a timelike point, induces the notion of orientation reversion given by the involution $\sigma_{\boldsymbol{p}}$. For another sphere complex $\mathcal{L} \cap \boldsymbol{q}^\perp$, where $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+2}$, to play the distinguished role of the set of "oriented hyperplanes" on $\mathcal{S}$ it must be invariant under orientation reversion, i.e., $\sigma_{\boldsymbol{p}}(\mathcal{L} \cap \boldsymbol{q}^\perp) = \mathcal{L} \cap \boldsymbol{q}^\perp$, which is equivalent to $\langle p, q \rangle = 0$.

**Definition 5.5.** For a point $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+2}$ with

$$\langle p, q \rangle = 0$$

we call the sphere complex

$$\mathcal{B} \coloneqq \mathcal{L} \cap \boldsymbol{q}^\perp,$$

a *plane complex.*

Up to a Lie transformation that fixes $\boldsymbol{p}$, i.e. a Möbius transformation (cf. Section 3.4), we can set, w.l.o.g.,

$$\boldsymbol{q} = \begin{cases} [e_{n+1}] = [0, \dots, 1, 0, 0] & \text{if } \langle q, q \rangle > 0 \\ [e_{n+2}] = [0, \dots, 0, 1, 0] & \text{if } \langle q, q \rangle < 0 \\ [e_\infty] = [0, \dots, \frac{1}{2}, \frac{1}{2}, 0] & \text{if } \langle q, q \rangle = 0. \end{cases}$$

Consider the restriction of the Lie quadric to $\boldsymbol{q}^\perp \simeq \mathbb{R}\mathrm{P}^{n+1}$. Then for the non-parabolic cases we identify each of the plane complexes with one of the *Laguerre quadrics* which we have introduced in Section 4. The parabolic plane complex corresponds to the classical case of Euclidean Laguerre geometry. Thus, we recover (see Figure 15)

- *hyperbolic Laguerre geometry* if $\langle q, q \rangle > 0$ (see Section 4.2),
- *elliptic ("spherical") Laguerre geometry* if $\langle q, q \rangle < 0$ (see Section 4.3),
- *Euclidean Laguerre geometry* if $\langle q, q \rangle = 0$ (see Section A.3).

*Remark* 5.4. Note that according to the classical naming convention of sphere complexes, which we adopted in Definition 5.4, an elliptic sphere complex is associated with hyperbolic Laguerre geometry, while a hyperbolic sphere complex is associated with elliptic Laguerre geometry.

The corresponding groups of *Laguerre transformations* are induced by the groups of Lie transformations that preserve the corresponding Laguerre quadric $\mathcal{B}$, or equivalently the point $\boldsymbol{q}$,

$$\mathbf{Lie}_{q}/_{\sigma_{\boldsymbol{q}}} \simeq \begin{cases} \mathrm{PO}(n, 2) & \text{if } \langle q, q \rangle > 0 \\ \mathrm{PO}(n+1, 1) & \text{if } \langle q, q \rangle < 0 \\ \mathrm{PO}(n, 1, 1) & \text{if } \langle q, q \rangle = 0, \end{cases}$$

where $\sigma_{\boldsymbol{q}}$ is the involution associated with the plane complex (cf. Definition 3.1), and we set $\sigma_{\boldsymbol{q}} = \mathrm{id}$ if $\langle q, q \rangle = 0$.

*Remark* 5.5. In the non-parabolic cases, the condition $\langle p, q \rangle = 0$ is equivalent to the condition that the two involutions $\sigma_{\boldsymbol{p}}$ and $\sigma_{\boldsymbol{q}}$ commute, i.e.

$$\sigma_{\boldsymbol{p}} \circ \sigma_{\boldsymbol{q}} = \sigma_{\boldsymbol{q}} \circ \sigma_{\boldsymbol{p}}.$$

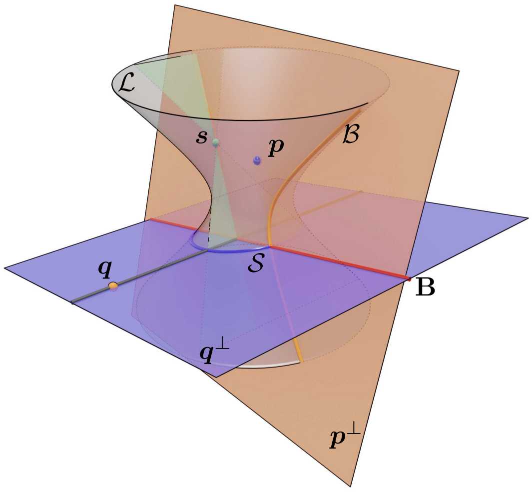



**Figure 15.** Laguerre geometry from Lie geometry. The choice of a point $\boldsymbol{q}$ with $\langle p, q\rangle = 0$ determines a plane complex, or Laguerre quadric $\mathcal{B} \subset \mathcal{L}$. This induces Laguerre geometry on a Cayley-Klein space in the base plane $\mathbf{B}$. The point $\boldsymbol{s} \in \mathcal{B}$ corresponds to an oriented line in that space via polar projection.

We have recognized the different Laguerre quadrics by their signature, which depends on the signature of the point $\boldsymbol{q}$ only but is entirely independent of the point $\boldsymbol{p}$ with $\langle p, q\rangle = 0$. We yet have to establish the geometric relation to Lie geometry.

**Definition 5.6.** Given the two points $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+2}$ defining the point complex and the plane complex respectively, we call the set

$$\mathbf{B} \coloneqq \boldsymbol{p}^\perp \cap \boldsymbol{q}^\perp \simeq \mathbb{R}\mathrm{P}^n.$$

the *base plane*.

In the restriction to the hyperplane of the point complex $\boldsymbol{p}^\perp$, the point $\pi_{\boldsymbol{p}}(\boldsymbol{q})$ plays the role of the point $\boldsymbol{q}$ from the Sections 4.2 and 4.3. Thus, polar projection with respect to this point yields hyperbolic/elliptic geometry in the base plane $\mathbf{B}$. In the parabolic case projection with respect to $\boldsymbol{q}$ should be replaced by stereographic projection, which recovers Euclidean (similarity) geometry (cf. Section A.2).

On the other hand, in the restriction to the hyperplane of the plane complex $\boldsymbol{q}^\perp$, the point $\pi_{\boldsymbol{q}}(\boldsymbol{p})$ plays the role of the point $\boldsymbol{p}$ from the Sections 4.2 and 4.3. Thus, polar projection with respect to this point yields hyperbolic/elliptic geometry in the base plane $\mathbf{B}$, while in the parabolic case projection with respect to $\boldsymbol{p}$ leads to dual Euclidean (similarity) geometry (cf. Section A.3).

On the level of the transformation group this can be described in the following way. Consider the Lie transformations $\mathbf{Lie}_{\boldsymbol{p},\boldsymbol{q}}$ that fix all "points" and "planes", i.e. the point complex $\mathcal{S}$ and the plane complex $\mathcal{B}$, or equivalently, the two points $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+2}$. These transformations naturally factor to ${}^{\mathbf{Lie}_{\boldsymbol{p},\boldsymbol{q}}}/_{\{\sigma_{\boldsymbol{p}}, \sigma_{\boldsymbol{q}}\}}$, where again we set $\sigma_{\boldsymbol{q}} = \mathrm{id}$ if $\langle q, q\rangle = 0$. Their action is well-defined on the quotient space ${}^{\mathcal{S}}/_{\sigma_{\boldsymbol{q}}}$, due to Remark 5.5. The quotient space ${}^{\mathcal{S}}/_{\sigma_{\boldsymbol{q}}}$ can be embedded into the base plane $\mathbf{B}$ using the projection $\pi_{\boldsymbol{q}}$, which should be replaced by stereographic projection in the Euclidean case (cf. Section A.3). Thus, we may equivalently consider the action of these Lie transformations on the base plane $\mathbf{B}$, on which they act as lower dimensional projective orthogonal groups again:

$$\mathbf{Lie}_{\boldsymbol{p},\boldsymbol{q}}\big/\{\sigma_{\boldsymbol{p}}, \sigma_{\boldsymbol{q}}\} \simeq \mathrm{PO}(n+1,1)\big/\sigma_{\boldsymbol{q}} \simeq \begin{cases} \mathrm{PO}(n,1), & \text{if } \langle q, q\rangle > 0 \\ \mathrm{PO}(n+1), & \text{if } \langle q, q\rangle < 0 \\ \mathrm{PO}(n,0,1), & \text{if } \langle q, q\rangle = 0. \end{cases}$$

We recognize $\mathrm{PO}(n,1)$ and $\mathrm{PO}(n+1)$ as the isometry groups of hyperbolic and elliptic space (cf. Sections 2.4 and 2.5), while $\mathrm{PO}(n,0,1)$ corresponds to the group of dual similarity transformations, i.e. the group of dual transformations $\mathrm{PO}(n,0,1)^*$ corresponds to isometries and scalings of Euclidean space (cf. Section A.1).

*Remark* 5.6. We end up with two models of the space form associated to each Laguerre geometry (see Figure 15). One is represented by the point complex $\mathcal{S} \subset \boldsymbol{p}^\perp \simeq \mathbb{R}\mathrm{P}^{n+1}$, with opposite points with respect to $\sigma_{\boldsymbol{q}}$ identified, which we refer to as the *spherical model.* In this model the oriented hyperspheres that correspond to sections of $\mathcal{S}$ with hyperplanes that contain the point $\pi_{\boldsymbol{p}}(\boldsymbol{q})$ are the distinguished "oriented hyperplanes"

Another model is obtained by its projection $\pi_{\boldsymbol{q}}(\mathcal{S})$ onto the base plane $\mathbf{B} \simeq \mathbb{R}\mathrm{P}^n$, which we refer to as the *projective model.* In this model the "oriented hyperplanes" become (oriented) projective hyperplanes.

**Proposition 5.3.**

*(i) In the non-Euclidean cases of Laguerre geometry, i.e. $\langle q,q\rangle \neq 0$, the point complex $\mathcal{S}$ may be identified with hyperbolic / elliptic space respectively, after taking the quotient with respect to $\sigma_{\boldsymbol{q}}$, or equivalently, projection onto the base plane*

$$\mathcal{S}/_{\sigma_{\boldsymbol{q}}} \simeq \pi_{\boldsymbol{q}}(\mathcal{S}) \subset \mathbf{B} \simeq \mathbb{R}\mathrm{P}^n.$$

*The Lie transformations that fix the point complex and the plane complex act on $\pi_{\boldsymbol{q}}(\mathcal{S}) \subset \mathbf{B}$ as the corresponding isometry group.*

*(ii) In the case of Euclidean Laguerre geometry, i.e. $\langle q,q\rangle = 0$, the point complex $\mathcal{S}$ may be identified with Euclidean space upon stereographic projection. The Lie transformations that fix the point complex and the plane complex act on $\mathbf{B}$ as dual similarity transformations.*

*Remark* 5.7. In Laguerre geometry the hyperplanar sections correspond to oriented spheres, which, in the non-Euclidean cases, can be identified with their polar points. In elliptic Laguerre geometry the Lie quadric projects to the "outside" of the elliptic Laguerre quadric

$$\pi_{\boldsymbol{q}}(\mathcal{L}) = \mathcal{B}^+_{\mathrm{ell}} \cup \mathcal{B}_{\mathrm{ell}}$$

which represents all poles of hyperplanar sections (9). In hyperbolic Laguerre geometry, on the other hand, the Lie quadric projects to the "inside" of the hyperbolic Laguerre quadric

$$\pi_{\boldsymbol{q}}(\mathcal{L}) = \mathcal{B}^-_{\mathrm{hyp}} \cup \mathcal{B}_{\mathrm{ell}},$$

while the poles of hyperplanar sections are the whole space (8). The Lie quadric only projects to the points corresponding to hyperbolic spheres/distance hypersurfaces/horospheres, and not to points representing deSitter spheres (cf. Remark 4.5 (i)). Vice versa, Laguerre spheres that are deSitter spheres do not possess a (real) lift to the Lie quadric.

### 5.3 Subgeometries of Lie geometry

Choosing different signatures for the points $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e. different signatures for the point complex and plane complex, we recover different subgeometries of Lie geometry (see Table 6).

Fixing both points in the Lie group induces (a quadruple covering of) the corresponding isometry group. We call the group obtained by fixing only $\boldsymbol{p}$ (a double cover of) the corresponding *Möbius group*, and the group obtained by fixing only $\boldsymbol{q}$ (a double cover of) the corresponding *Laguerre group.* For each isometry group the corresponding Möbius group describes the transformations that map points in the space form to points while preserving spheres, while the Laguerre group describes the transformations that map hyperplanes to hyperplanes while preserving spheres. For this to hold the transformations either have to be considered locally, or acting on the set of oriented points / oriented hyperplanes respectively (see Theorem 3.8, Remark 3.9, Remark 3.13 (iii), Remark 3.14 (iii), Section 4.2.1, Section 4.3.1).

*Remark* 5.8. Note that certain geometries have the same transformation group. In particular does $n$-dimensional Lie geometry have the same transformation group as $(n+1)$-dimensional hyperbolic Laguerre geometry. Geometrically this is due to the fact that one may identify the oriented hyperspheres of $\mathcal{S}$ with the oriented hyperbolic hyperplanes of the inside $\mathcal{H} = \mathcal{S}^{-}$.

| **space form** | **isometry grp.** | **Möbius grp.** | **Laguerre grp.** | **sign. $\boldsymbol{p}$, $\boldsymbol{q}$** |
|---|---|---|---|---|
| elliptic space | $\mathrm{PO}(n+1)$ | $\mathrm{PO}(n+1,1)$ | $\mathrm{PO}(n+1,1)$ | $(-)$ $(-)$ |
| hyperbolic space | $\mathrm{PO}(n,1)$ | $\mathrm{PO}(n+1,1)$ | $\mathrm{PO}(n,2)$ | $(-)$ $(+)$ |
| deSitter space | $\mathrm{PO}(n,1)$ | $\mathrm{PO}(n,2)$ | $\mathrm{PO}(n+1,1)$ | $(+)$ $(-)$ |
| (dual) Euclidean space | $\mathrm{PO}(n,0,1)$ | $\mathrm{PO}(n+1,1)$ | $\mathrm{PO}(n,1,1)$ | $(-)$ $(0)$ |
| (dual) Minkowski space | $\mathrm{PO}(n-1,1,1)$ | $\mathrm{PO}(n,2)$ | $\mathrm{PO}(n,1,1)$ | $(+)$ $(0)$ |

**Table 6.** Isometry group, Möbius group, and Laguerre group for different space forms, and the signatures of the points $\boldsymbol{p}$ and $\boldsymbol{q}$ defining the corresponding point complex and plane complex in Lie geometry respectively. In the degenerate cases of Euclidean and Minkowski geometry, the given "isometry group" is actually the similarity group on the dual space.

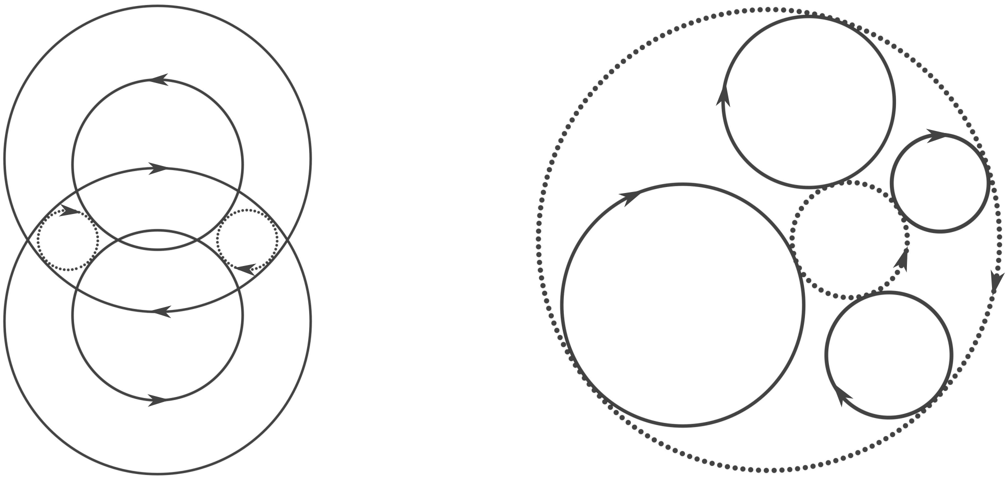


**Figure 16.** Lie-circumscribed quadrilaterals.

# 6 Checkerboard incircular nets

Let $n = 2$. As an application of two-dimensional Lie and Laguerre geometry we study the properties of checkerboard incircular nets in these geometries.

## 6.1 Checkerboard incircular nets in Lie geometry

To investigate configurations of oriented circles and their oriented contact on the two-sphere, we identify oriented circles with points on the Lie quadric $\mathcal{L} \subset \mathbb{R}\mathrm{P}^4$, which is a quadric of signature $(+++--)$, as described in Section 5.

**Definition 6.1** (Lie quadrilateral)**.** A *Lie quadrilateral* is a quadruple of oriented circles, called *edge circles*.

*Remark* 6.1. Two edge circles of a Lie quadrilateral do not necessarily intersect. Thus all quadrilaterals shown in Figure 16 are admissible Lie quadrilaterals.

**Definition 6.2** (Lie circumscribed)**.** A Lie quadrilateral is called *circumscribed* if the four points on the Lie quadric corresponding to its four oriented edge circles are coplanar. We call the signature of the plane in which these points lie the *signature* of the circumscribed Lie quadrilateral.

To justify the term "circumscribed" consider a plane $\mathbf{U} \subset \mathbb{R}\mathrm{P}^4$ of signature $(++-)$. Then according to Lemma 1.5 its polar line has signature $(+-)$, and thus, $\mathbf{U}^\perp \cap \mathcal{L} = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ consists of exactly two points. The one parameter family of circles corresponding to the points in $\mathbf{U} \cap \mathcal{L}$ are the circles in oriented contact with the two circles corresponding to $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Therefore, a circumscribed Lie quadrilateral of signature $(++-)$ is in oriented contact with exactly two circles (see Figure 16).

To characterize all possible cases of circumscribed Lie quadrilaterals we need to distinguish all possible signatures of the plane $\mathbf{U}$.

**Proposition 6.1.** *For a plane $\mathbf{U} \subset \mathbb{R}\mathrm{P}^4$ the family of oriented circles corresponding to $\mathbf{U} \cap \mathcal{L}$ is exactly one of the following depending on the signature of $\mathbf{U}$ with respect to the Lie quadric $\mathcal{L}$.*

- $(+++)$ *Empty family.*
- $(++-)$ *One parameter family of circles in oriented contact with the two oriented circles given by $\mathbf{U} \cap \mathcal{L}$.*

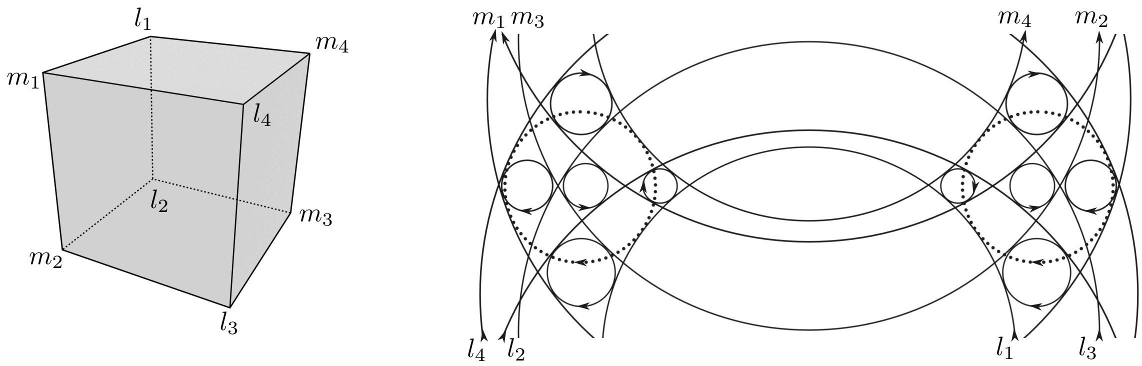



**Figure 17.** Lie geometric version of Miquel's theorem. *Left:* Combinatorial picture. *Right:* Geometric picture.

- $(+--)$ *Circles from the intersection of two hyperbolic circle complexes (cf. Definition 5.4 and Section B.2)*
- $(+-0)$ *Two contact elements (see Definition 5.2) with a common circle.*
- $(++0)$ *Exactly one circle.*
- $(+00)$ *One contact element.*

*Proof.* The Lie quadric has signature $(+++--)$. Thus, the listed signatures are all possible cases that can occur. A plane with signature $(+++)$ does not intersect the Lie quadric. The case $(++-)$ was already discussed before the proposition. For the case $(+--)$ the polar line has signature $(++)$. Thus, we may view $\mathbf{U}$ as the intersection of two hyperbolic circle complexes. The cases $(+-0)$, $(++0)$, and $(+00)$ each describes a tangent plane that, in turn, intersects the Lie quadric in two isotropic subspaces, touches the Lie quadric in exactly one point, intersects the Lie quadric in exactly one isotropic subspace. □

*Remark* 6.2. For the generic case of a circumscribed Lie quadrilateral, in the sense that the four points on the Lie quadric corresponding to its four edge circles span a plane, or equivalently no three of them are collinear, only the signatures $(++-)$, $(+--)$, and $(+-0)$ can occur.

The definition of Lie circumscribility via planarity in the Lie quadric immediately implies a Lie geometric version of the classical Miquel's theorem. To see this, we employ the following statement of projective geometry about the eight intersection points of three quadrics in space. [BS2008, Theorem 3.12]

**Lemma 6.2** (Associated points)**.** *Given eight distinct points which are the set of intersections of three quadrics in $\mathbb{RP}^3$, all quadrics through any seven of those points must pass through the eighth point.*

**Theorem 6.3** (Miquel's theorem in Lie geometry)**.** *Let $l_1, l_2, l_3, l_4, m_1, m_2, m_3, m_4$ be eight generic oriented circles on the sphere. If the five Lie quadrilaterals $(l_1, l_2, m_1, m_2)$, $(l_1, l_2, m_3, m_4)$, $(l_3, l_4, m_1, m_2)$, $(l_3, l_4, m_3, m_4)$, $(l_2, l_3, m_2, m_3)$ are circumscribed, then so is the quadrilateral $(l_1, l_4, m_1, m_4)$ (see Figure 17).*

*Remark* 6.3. A sufficient genericity condition for the eight points on the Lie quadric is that no three are collinear and no five coplanar.

*Proof.* Consider the eight points on the Lie quadric as the vertices of a combinatorial cube (see Figure 17). Coplanarity of the bottom and side faces corresponds to the assumed circumscribility. Thus, we have to show that the top face is planar as well.

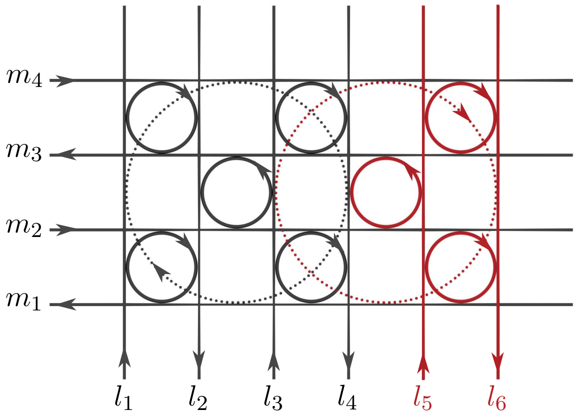



**Figure 18.** On the combinatorics of adjacent “cubes” of a checkerboard incircular net.

As a first step we show that all eight vertices of the cube are contained in a three-dimensional projective subspace. Indeed, let $V$ be the subspace spanned by $l_2, l_3, m_1, m_2$. Then the assumed circumscribility implies that for instance $l_1$ lies in a plane with $l_2, m_1, m_2$ and therefore $l_1 \in V$. Similarly, $l_4, m_3 \in V$, and finally $m_4 \in V$.

A three-dimensional subspace intersects the Lie quadric in a (at most once degenerate) two-dimensional quadric $\tilde{\mathcal{L}}$. Consider the two degenerate quadrics $\mathcal{Q}_1, \mathcal{Q}_2$ consisting of two opposite face planes of the cube, respectively. Then, due to the genericity condition, the eight points of the cube are the intersection points of $\tilde{\mathcal{L}}, \mathcal{Q}_1, \mathcal{Q}_2$. Now consider the degenerate quadric $\mathcal{Q}_3$ consisting of the bottom plane of the cube and the plane spanned by $l_1, l_4, m_1$. Then $\mathcal{Q}_3$ contains seven of the eight points, and therefore, according to Lemma 6.2, also the eighth point $m_4$. Since $m_4$ may not lie in the bottom plane, we conclude that the quadrilateral $(l_1, l_4, m_1, m_4)$ is circumscribed. □

We now introduce nets consisting of two families of oriented circles such that every second Lie quadrilateral (in a checkerboard-manner) is circumscribed.

**Definition 6.3** (Lie checkerboard incircular nets)**.** Two families $(l_i)_{i=1}^{\infty}$, $(m_i)_{i=1}^{\infty}$ of oriented circles on the sphere are called a *Lie checkerboard incircular net* if for every $i, j \in \mathbb{Z}$ with even $i + j$ the Lie quadrilateral $(l_i, l_{i+1}, m_j, m_{j+1})$ is circumscribed.

In the following we will always assume generic Lie checkerboard incircular nets in the sense of Remark 6.3. As an immediate consequence of Theorem 6.3 we find that Lie checkerboard incircular nets have more circumscribed Lie quadrilaterals than introduced in its definition.

**Corollary 6.4.** *Let $(l_i)_{i=1}^{\infty}$, $(m_i)_{i=1}^{\infty}$ be the oriented circles of a Lie checkerboard incircular net. Then for every $i, j, k \in \mathbb{Z}$ with even $i + j$ the Lie quadrilateral $(l_i, m_j, l_{i+2k+1}, m_{j+2k+1})$ is circumscribed.*

Similar to the argument in the proof of Theorem 6.3 (or as a consequence thereof), we find that the points on the Lie quadric corresponding to a Lie checkerboard incircular net can not span the entire space.

**Theorem 6.5.** *The points on the Lie quadric $\mathcal{L} \subset \mathbb{RP}^4$ corresponding to the oriented circles of a Lie checkerboard incircular net lie in a common hyperplane of $\mathbb{RP}^4$.*

*Proof.* Consider “adjacent” cubes $(l_1, l_2, l_3, l_4, m_1, m_2, m_3, m_4)$ and $(l_3, l_4, l_5, l_5, m_1, m_2, m_3, m_4)$ from the Lie checkerboard incircular net with vertices on the Lie quadric (see Figure 18). Each of these cubes lies in a three-dimensional subspace of $\mathbb{RP}^4$, and they coincide in six of its eight vertices. Thus, both cubes, and by induction the whole net, lie in the same three-dimensional subspace. □

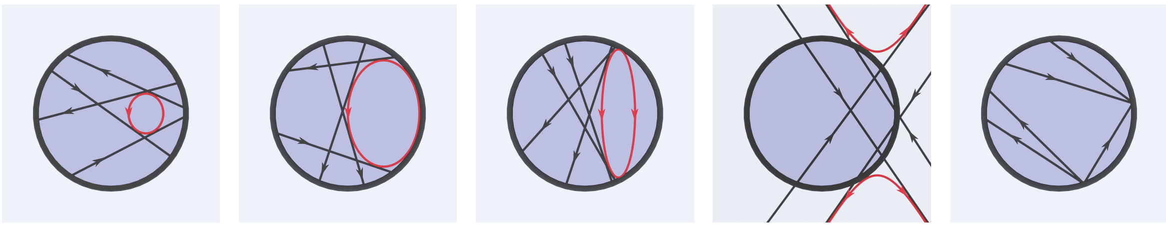


**Figure 19.** Inscribed quadrilaterals in the hyperbolic plane. The right most case is degenerate and consists of four oriented lines "touching" an oriented line at its points at infinity.

As we have seen in Section 4, depending on its signature, a three-dimensional subspace of the Lie quadric induces one of the three types of Laguerre geometry. Thus, our study of Lie checkerboard incircular nets may be reduced to the study of its three Laguerre geometric counterparts as we will see in the next section.

### 6.2 Laguerre checkerboard incircular nets

Two dimensional hyperbolic/elliptic/Euclidean Laguerre geometry is the geometry of oriented lines in the hyperbolic/elliptic/Euclidean plane and their oriented contact to generalized oriented circles (Laguerre circles) in the respective space form. We identify oriented lines with points on, and oriented circles with planar sections of, the corresponding Laguerre quadric $\mathcal{B}$, which is a quadric of signature $(++--)$, $(+++-)$, $(++-0)$ respectively (see Sections 4.2, 4.3, A.3).

Similar to the condition for Lie circumscribility, four oriented lines touch a common oriented circle if and only if the corresponding points on the Laguerre quadric are coplanar. On the other hand, all three Laguerre geometries are subgeometries of Lie geometry, by restricting the Lie quadric to a three-dimensional subspace. Thus, in this restriction a Lie circumscribed quadrilateral turns into four lines touching a common oriented circle. Accordingly one obtains the following Laguerre geometric version of Theorem 6.3.

**Theorem 6.6** (Miquel's theorem in Laguerre geometry)**.** *Let $l_1, l_2, l_3, l_4, m_1, m_2, m_3, m_4$ be eight generic oriented lines in the hyperbolic/elliptic/Euclidean plane. If the five quadrilaterals $(l_1, l_2, m_1, m_2)$, $(l_1, l_2, m_3, m_4)$, $(l_3, l_4, m_1, m_2)$, $(l_3, l_4, m_3, m_4)$, $(l_2, l_3, m_2, m_3)$ are circumscribed (each touches a common oriented circle), then so is the quadrilateral $(l_1, l_4, m_1, m_4)$ (cf. Figure 18).*

The Laguerre geometric version of checkerboard incircular nets (see Definition 6.3) is the following.

**Definition 6.4** (Checkerboard incircular nets in a space form)**.** Two families $(l_i)_{i=1}^{\infty}$, $(m_i)_{i=1}^{\infty}$ of oriented lines in the hyperbolic/elliptic/Euclidean plane are called a *(hyperbolic/elliptic/Euclidean) checkerboard incircular net* if for every $i, j \in \mathbb{Z}$ with even $i + j$ the four lines $l_i, l_{i+1}, m_j, m_{j+1}$ touch a common oriented circle (Laguerre circle).

*Remark* 6.4. From Corollary 6.4, or Theorem 6.6, we find that, same as in the Lie geometric case, every quadrilateral $(l_i, m_j, l_{i+2k+1}, m_{j+2k+1})$, $i, j, k \in \mathbb{Z}$ of a checkerboard incircular net with even $i + j$ is circumscribed.

Now we can formulate the following classification result for Lie checkerboard incircular nets.

**Theorem 6.7** (classification of Lie checkerboard incircular nets)**.** *Every Lie checkerboard incircular net is given by a Lie transformation of a hyperbolic, elliptic, or Euclidean checkerboard incircular net.*

*Proof.* According to Theorem 6.5 every Lie checkerboard incircular net lies in a three-dimensional subspace of $\mathbb{RP}^4$. This subspace can only have one of the signatures $(+++-)$, $(++--)$, $(++-0)$

and thus may be identified (after a certain Lie transformation) with a checkerboard incircular net in the corresponding Laguerre geometry. □

In the different space forms different types of generic (see Remark 6.2) circumscribed quadrilaterals (see Proposition 6.1) can still occur (cf. Remark 5.7):

▶ hyperbolic Laguerre geometry $(++--)$ (see Remark 4.5 (i) and Figure 19)

- $(++-)$: Four lines touching a common oriented hyperbolic sphere/distance curve/horosphere.
- $(+--)$: Four lines touching a common deSitter sphere.
- $(+-0)$: Four lines touching a common oriented hyperbolic line at infinity.

▶ elliptic Laguerre geometry $(+++-)$:

- $(++-)$: Four lines touching a common oriented elliptic sphere.

▶ Euclidean Laguerre geometry $(++-0)$:

- $(++-)$: Four lines touching a common oriented Euclidean sphere.
- $(+-0)$: Four lines from two families of parallel oriented lines.

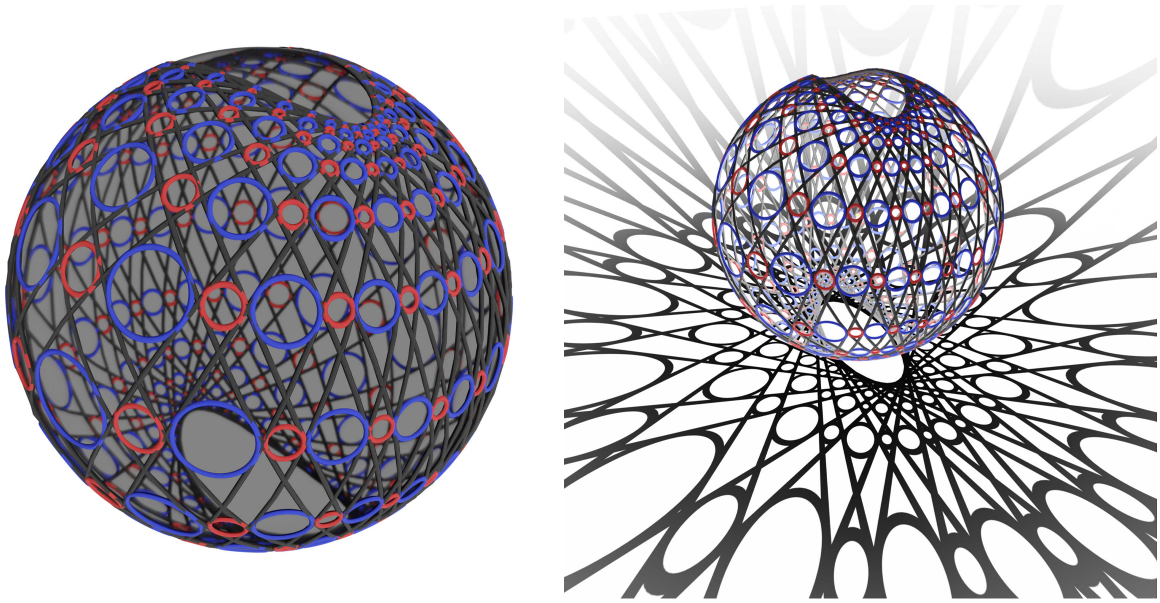


**Figure 20.** Elliptic checkerboard incircular net on the Möbius quadric (*left*) and the projection from its center to the elliptic plane (*right*).

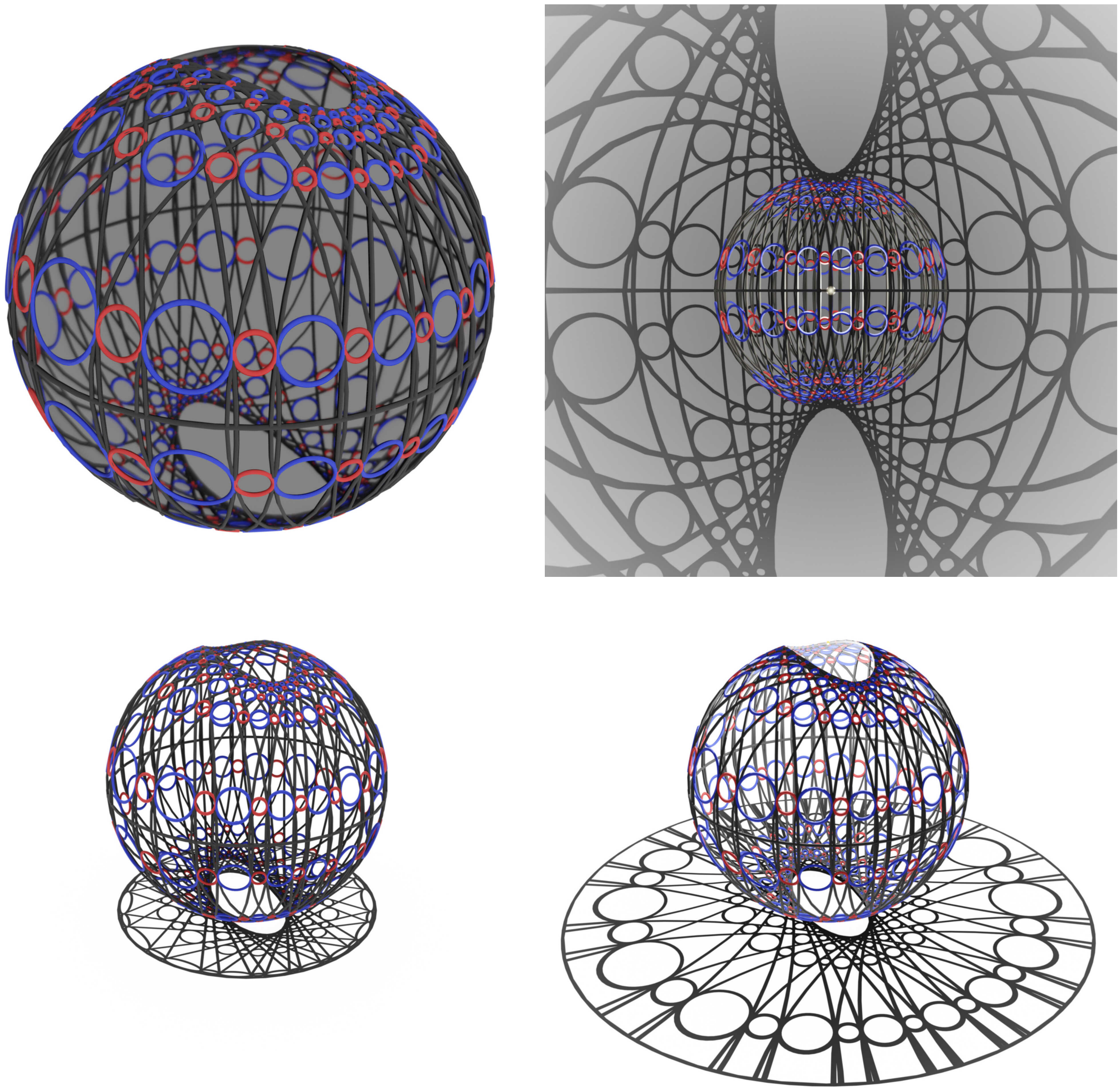


**Figure 21.** Hyperbolic checkerboard incircular net on the Möbius quadric (*top-left*). Projection to the Klein-Beltrami disk (*bottom-left*), the Poincaré disk (*bottom-right*) and the Poincaré half plane (*top-right*).

### 6.3 Hypercycles

In Laguerre geometry the oriented lines, and not the points, of a given space form are invariant objects. Thus, in Laguerre geometry, it is natural to describe an (oriented) curve in the (hyperbolic/elliptic/Euclidean) plane by its (oriented) tangent lines.[1] Conversely we say that every curve on the Laguerre quadric corresponds to a curve in the plane. Note that in the case of the hyperbolic plane the envelope of such a "curve" might lie partially (or even entirely) "outside" the hyperbolic plane. We still consider this to be an admissible (non-empty) *Laguerre curve*.

**Definition 6.5.** The one-parameter family of oriented lines (in the hyperbolic/elliptic/Euclidean plane) corresponding to a curve on the Laguerre quadric $\mathcal{B}$ is called a *(hyperbolic/elliptic/Euclidean) Laguerre curve*.

We have noted that planar sections of the Laguerre quadric correspond to Laguerre circles,

[1] The classical german terminology for this description is "Klassenkurve" [Kle1928].

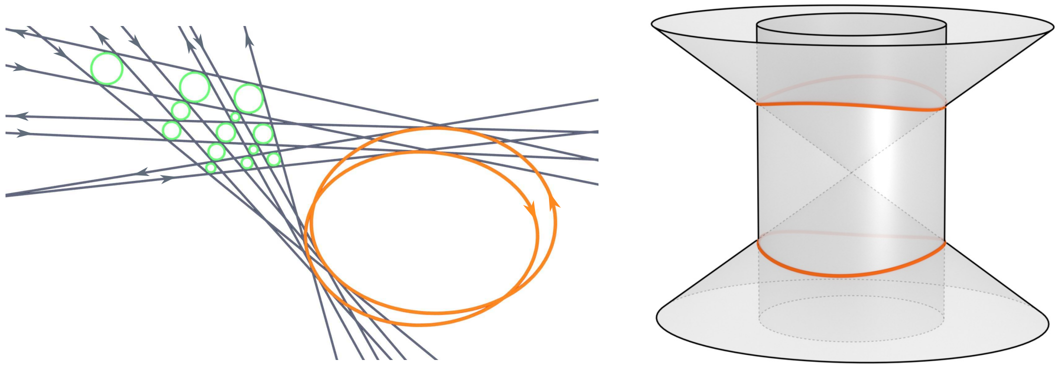


**Figure 22.** Hypercycle in the Euclidean plane, and its corresponding hypercycle base curve on $\mathcal{B}$.

also called (generalized) *cycles* in the two-dimensional case. Consequently, the next higher order intersections with the Laguerre quadric are called *hypercycles* [Bla1910].

**Definition 6.6.** A (hyperbolic/elliptic/Euclidean) Laguerre curve corresponding to the intersection of the Laguerre quadric with another quadric is called a *(hyperbolic/elliptic/Euclidean) hypercycle*. The corresponding curve on the Laguerre quadric is called *hypercycle base curve*.

**Example 6.1.** A conic endowed with both orientations, joined together as two components of a single oriented curve (see Section 6.4) is a hypercycle in every space form. A more generic example is shown in Figure 22.

The intersection curve of two quadrics (*base curve*) is contained in all quadrics of the pencil spanned by the two quadrics. Thus, a hypercycle, through its hypercycle base curve, corresponds not just to one quadric but the whole pencil of quadrics spanned by it and the Laguerre quadric.

The following theorem establishes a relation between a checkerboard incircular net and a hypercycle, as well as two certain hyperboloids in the pencil of quadrics corresponding to its hypercycle base curve. In the Euclidean case this was shown in [BST2018, Theorem 3.4] as part of an incidence theorem for checkerboard incircular nets (see Theorem 6.15). A proof of Lemma 6.9 can also be found there.

**Theorem 6.8.** *The lines of a (hyperbolic/elliptic/Euclidean) checkerboard incircular net are in oriented contact with a common hypercycle (see Figure 22).*

*Moreover, the corresponding pencil of quadrics, which contains the hypercycle base curve, contains two unique hyperboloids $\mathcal{Q}, \widetilde{\mathcal{Q}}$ distinguished in the following way (see Figure 23). Let $(l_i)_{i=1}^{\infty}$, $(m_i)_{i=1}^{\infty}$ be the points on the Laguerre quadric $\mathcal{B} \subset \mathbb{RP}^3$ corresponding to the oriented lines of the checkerboard incircular net. Consider the lines $L_i := (l_i, l_{i+1})$, and $M_i := (m_i, m_{i+1})$. Then, all lines $L_{2k}$, $M_{2l}$ lie on a common hyperboloid $\mathcal{Q} \subset \mathbb{RP}^3$. Similarly, all lines $L_{2k+1}$, $M_{2l+1}$ lie on a common hyperboloid $\widetilde{\mathcal{Q}} \subset \mathbb{RP}^3$.*

*Proof.* Due to the inscribability property of checkerboard incircular nets every line $L_{2k}$ intersects every line $M_{2l}$, and vice versa. Thus, all lines $L_{2k}$, $M_{2l}$ generically lie on a common hyperboloid $\mathcal{Q}$. Similarly, all lines $L_{2k+1}$, $M_{2l+1}$ lie on a common hyperboloid $\widetilde{\mathcal{Q}}$. We now show that both hyperboloids $\mathcal{Q}$, $\widetilde{\mathcal{Q}}$ intersect the Laguerre quadric $\mathcal{B}$ in the same curve, that is, they belong to the same pencil of quadrics. Indeed, according to Lemma 6.9, for each line $L_{2k+1}$, there exists a unique quadric in the pencil spanned by $\mathcal{B}$ and $\mathcal{Q}$ containing $L_{2k+1}$. Same for each line $M_{2l+1}$. Since the lines $L_{2k+1}$ and $M_{2l+1}$ pairwise intersect, again according to Lemma 6.9, the corresponding quadrics coincide with each other and eventually with $\widetilde{\mathcal{Q}}$. Thus, all points $l_i$, $m_j$ lie on the intersection $\mathcal{B} \cap \mathcal{Q} = \mathcal{B} \cap \widetilde{\mathcal{Q}}$. □

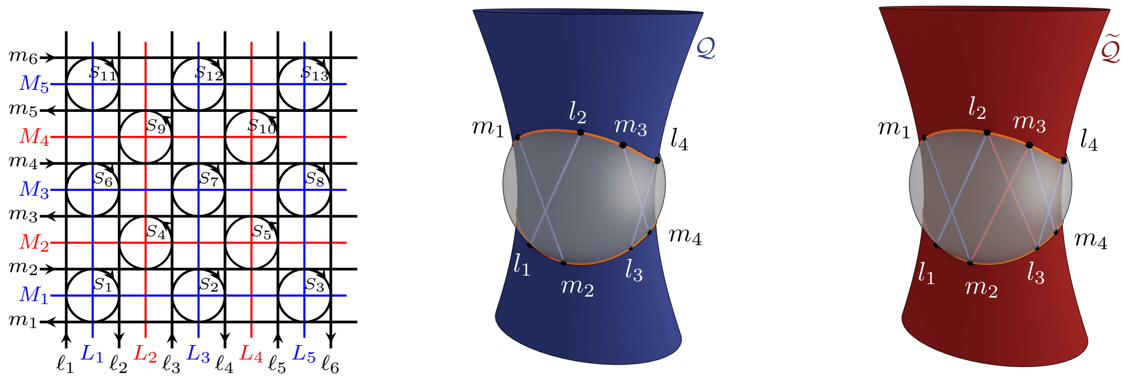



**Figure 23.** *Left:* Combinatorial picture of the lines of a checkerboard incircular net. *Middle/Right:* The two hyperboloids in the pencil of quadrics through the hypercycle base curve associated with a checkerboard incircular net in the elliptic plane.

**Lemma 6.9.** *Let $p_1, p_2$ be two points which belong to all members of a pencil of quadrics $Q_t$. Then, there exists a unique quadric $Q_{t_{12}}$ from the pencil which contains the whole line $L_{12} = (p_1, p_2)$. If the line $L_{34} = (p_3, p_4)$ associated with another pair of base points $p_3, p_4$ intersects the line $L_{12}$ then the two quadrics $Q_{t_{12}}$ and $Q_{t_{34}}$ coincide.*

The oriented circles of a checkerboard incircular net correspond to the planes spanned by pairs of lines $L_{2k}, M_{2l}$ or $L_{2k+1}, M_{2l+1}$, i.e. they correspond to tangent planes of the two hyperboloids $\mathcal{Q}, \widetilde{\mathcal{Q}}$, respectively. We identify each circle with its polar point with respect to the Laguerre quadric $\mathcal{B}$, or in the Euclidean case with a point in the cyclographic model (cf. Section A.3).

**Corollary 6.10.**

*(i)* *The polar points corresponding to the oriented circles of a hyperbolic/elliptic checkerboard incircular net lie on two quadrics, the polar pencil of which contains the Laguerre quadric (polar with respect to the Laguerre quadric).*

*(ii)* *The points in the cyclographic model corresponding to the oriented circles of a Euclidean checkerboard incircular net lie on two quadrics, the dual pencil of which contains the absolute quadric (i.e. the two quadrics are Minkowski confocal quadrics).*

*Proof.*

(i) Under polarization in the Laguerre quadric $\mathcal{B}$ the tangent planes of $\mathcal{Q}$ become points on the polar quadric $\mathcal{Q}^{\perp}$. Similarly, the tangent planes of $\widetilde{\mathcal{Q}}$ become points on the polar quadric $\widetilde{\mathcal{Q}}^{\perp}$. Since $\mathcal{Q}$, $\widetilde{\mathcal{Q}}$, and $\mathcal{B}$ are contained in a common pencil of quadrics, their polar images $\mathcal{Q}^{\perp}$, $\widetilde{\mathcal{Q}}^{\perp}$, and $\mathcal{B}^{\perp} \cong \mathcal{B}$ are contained in the polar pencil of quadrics.

(ii) For the Euclidean case a similar argument holds under dualization to the cyclographic model.

□

*Remark* 6.5 (cf. [Böh1970, Sau1925]). According to Remark 6.4 a checkerboard incircular net possesses more incircles than imminent from its definition. If we collect the polar (or dual) points of all these circles they constitute the points of intersection of an octahedral-tetrahedral grid. The "diagonal surfaces" of this grid are quadrics from the polar (dual) pencil of quadrics from Corollary 6.10.

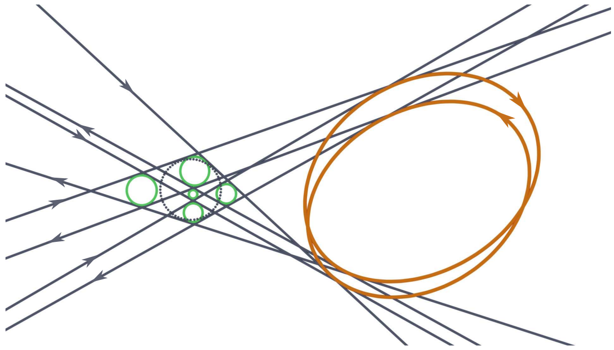


**Figure 24.** Incidence theorem for eight lines touching a hypercycle.

We conclude this section on hypercycles by stating an incidence result concerning eight lines touching a hypercycle, which is similar to Theorem 6.6.

**Theorem 6.11.** *Let $l_1, l_2, l_3, l_4, m_1, m_2, m_3, m_4$ be eight generic lines touching a hypercycle. If the three quadrilaterals $(l_1, l_2, m_1, m_2)$, $(l_2, l_3, m_2, m_3)$, $(l_3, l_4, m_3, m_4)$ are circumscribed, then so is the quadrilateral $(l_1, l_4, m_1, m_4)$ (see Figure 24).*

*Proof.* We identify the eight oriented lines with its corresponding points on the Laguerre quadric. The hypercycle base curve is the intersection of two quadrics. Define the degenerate quadric given by the two planes through $l_1, l_2, m_1, m_2$ and $l_3, l_4, m_3, m_4$ respectively. Then the given eight points on the Laguerre quadric are the intersection of those three quadrics. According to Lemma 6.2 every quadric through seven of those points must pass through the eighth. Consider the degenerate quadric given by the two planes through $l_2, l_3, m_2, m_3$ and $l_1, l_4, m_1$ respectively. Then this quadric must also pass through $m_4$. Since no five points may lie in a plane we can conclude that $l_1, l_4, m_1, m_4$ lie in a common plane, and thus, that the corresponding quadrilateral is circumscribed. □

## 6.4 Conics and incircular nets

Towards the classification and parametrization of checkerboard incircular nets it turns out to be useful to consider certain normal forms of hypercycles, one of which are conics. In the projective model of of the hyperbolic/elliptic/Euclidean plane (Cayley-Klein spaces, see Section 2) (metric) conics are given by projective conics.

**Definition 6.7** (Conics in spaceforms)**.** In the projective model of the hyperbolic/elliptic/Euclidean plane embedded into $\mathbb{RP}^2$ a *(hyperbolic/elliptic/Euclidean) conic* is a conic in $\mathbb{RP}^2$.

*Remark* 6.6. In hyperbolic geometry the conic might lie "outside" the hyperbolic plane and be considered a "deSitter conic".

**Lemma 6.12.** *For a hypercycle in the hyperbolic/elliptic/Euclidean plane the following three statements are equivalent*

*(i) The hypercycle base curve is given by the intersection of the Laguerre quadric with a cone with vertex $\boldsymbol{p}$.*

*(ii) The hypercycle consists of two components that coincide up to their orientation.*

*(iii) The hypercycle is a conic (doubly covered with opposite orientation).*

*Proof.* The hypercycle base curve is invariant under the involution $\sigma_{\boldsymbol{p}}$ reversing the orientation, if and only if $\boldsymbol{p}$ is the vertex of a cone intersecting the Laguerre quadric in the hypercycle base curve.

In hyperbolic and elliptic Laguerre geometry $\boldsymbol{p}$ is the polar point of the base plane of the projective model of the corresponding space form. The polar of a cone with vertex $\boldsymbol{p}$ is therefore a conic contained in this base plane. Thus the tangent planes to the hypercycle base curve are the planes tangent to a conic, if and only if $\boldsymbol{p}$ is the vertex of a cone intersecting the Laguerre quadric in the hypercycle base curve. In that case corresponding oriented lines envelop the conic (twice with opposite orientations).

In Euclidean Laguerre geometry a similar argument holds upon dualization and considering the cyclographic model. □

*Remark* 6.7. The image of the set of (doubly covered) conics under all Laguerre transformations in a space form is open in the set of all hypercycles in that space form.

In the limit in which all incircles of the quadrilaterals $l_{2k}, l_{2k+1}, m_{2l}, m_{2l+1}$ of a checkerboard incircular net collapse to a point, the pairs of lines $l_{2k}, l_{2k+1}$ as well as the pairs of lines $m_{2l}, m_{2l+1}$ coincide respectively up to their orientation. Such a pair of oriented lines may be regarded as a non-oriented line. The points on the Laguerre quadric corresponding to two lines that agree up to their orientation are connected by a line that goes through the point $\boldsymbol{p}$. Considering the associated hyperboloids of a checkerboard incircular net from Theorem 6.8 we find that the generator lines $L_{2k}$, $M_{2l}$ all go through the point $\boldsymbol{p}$ and the hyperboloid $\mathcal{Q}$ becomes a cone with vertex at $\boldsymbol{p}$. In this limit a checkerboard incircular net becomes an "ordinary" incircular net.

**Definition 6.8.** Two families $(l_i)_{i=1}^{\infty}$, $(m_i)_{i=1}^{\infty}$ of (non-oriented) lines in the hyperbolic/elliptic/Euclidean plane are called a *(hyperbolic/elliptic/Euclidean) incircular net* (IC-net) if for every $i, j \in \mathbb{Z}$ the four lines $l_i, l_{i+1}, m_j, m_{j+1}$ touch a common circle (non-oriented Laguerre circle).

Now Lemma 6.12 allows for a characterization of incircular nets among the set of checkerboard incircular nets in terms of the type of its two associated hyperboloids (see Theorem 6.8).

**Theorem 6.13.** *A checkerboard incircular net is an incircular net, if one of its two associated hyperboloids is a cone with vertex at* $\boldsymbol{p}$.

While checkerboard incircular nets are instances of the corresponding (hyperbolic/elliptic/Euclidean) Laguerre geometry, incircular nets are a notion of the corresponding metric geometry, i.e. only invariant under isometries. The touching hypercycle becomes a touching conic.

**Theorem 6.14.** *All lines of a (hyperbolic/elliptic/Euclidean) incircular net touch a common conic.*

*Proof.* This follows from Theorem 6.8, Lemma 6.12, and Theorem 6.13. □

*Remark* 6.8. The intersection points of an incircular net lie on conics which are confocal with the touching conic. This property follows from the observations in Remark 6.5 (see [Böh1970] for the Euclidean case). It is closely related to the classical theorem of Graves-Chasels (see [Böh1970], and [AB2018] for the Euclidean and hyperbolic case), which holds in all three space forms (see [Izm2017]).

For the purpose of parametrization it turns out to be convenient to introduce the class of confocal checkerboard incircular nets, which lies in between the classes of checkerboard incircular nets and incircular nets.

**Definition 6.9.** A checkerboard incircular net is called *confocal checkerboard incircular net* if all its oriented lines touch a common conic.

### 6.5 Construction and parametrization of checkerboard incircular nets

The elementary construction of a checkerboard incircular net (line by line, while ensuring the incircle constraint) is guaranteed to work due to the following incidence theorem. [AB2018, BST2018]

**Theorem 6.15.** *Let $\ell_1, \cdots, \ell_6$, $m_1, \cdots, m_6$ be 12 oriented lines in the hyperbolic / elliptic / Euclidean plane which are in oriented contact with 12 oriented circles $S_1, \cdots, S_{12}$, in a checkerboard manner, as shown in Figure 23 (left). In particular, the lines $\ell_1, \ell_2, m_1, m_2$ are in oriented contact with the circle $S_1$, the lines $\ell_3, \ell_4, m_1, m_2$ are in oriented contact with the circle $S_2$ etc. Then, the 13th checkerboard quadrilateral also has an inscribed circle, i.e., the lines $\ell_5, \ell_6, m_5, m_6$ have a common circle $S_{13}$ in oriented contact.*

*Remark* 6.9. This incidence theorem holds in all three Laguerre geometries with literally the same proof as given in [BST2018] for the Euclidean case.

Though possible in principle, the elementary construction is not very stable and thus impractical for the construction of checkerboard incircular nets. Since all lines of a checkerboard incircular net are (oriented) tangent lines of a hypercycle, parametrizing the hypercycle, or equivalently parametrizing the hypercycle base curve, leads to another way of obtaining checkerboard incircular nets in terms of explicit formulas.

As the intersection of two quadrics the hypercycle base curve can be parametrized by elliptic functions. such parametrizations are given in [BST2018] in the case where the corresponding pencil of quadrics is diagonal, which corresponds to the case of the hypercycle being a (doubly covered) conic. This allows for the construction of Euclidean confocal checkerboard incircular nets, and, after applying a Laguerre transformation, all (generic) checkerboard incircular nets coming from hypercycles consisting of two components.

Projectively the base curve and corresponding pencils are all three Laguerre geometry. Thus, the same parametrizations as given in [BST2018] can be used for the hyperbolic and elliptic cases by reinterpreting another (non-degenerate) quadric of the pencil as the Laguerre quadric.

# A Euclidean cases

The Euclidean cases, which we have mainly omitted so far, are induced by degenerate quadrics. For a degenerate quadric $\mathcal{Q} \subset \mathbb{RP}^n$ polarity (see Section 1.3) does no longer define a bijection between the set of points and the set of hyperplanes. Instead one can apply the concept of duality.

The $n$-dimensional *dual real projective space* is given by

$$(\mathbb{RP}^n)^* := \mathrm{P}\left((\mathbb{R}^{n+1})^*\right),$$

where $\left(\mathbb{R}^{n+1}\right)^*$ is the space of linear functionals on $\mathbb{R}^n$. We identify $(\mathbb{RP}^n)^{**} = \mathbb{RP}^n$ in the canonical way, and obtain a bijection between projective subspaces $\boldsymbol{U} = \mathrm{P}(U) \subset \mathbb{RP}^n$ and their *dual projective subspaces*

$$\boldsymbol{U}^* := \mathrm{P}\left(\{\boldsymbol{y} \in (\mathbb{RP}^n)^* \mid \boldsymbol{y}(\boldsymbol{x}) = 0\}\right),$$

satisfying

$$\dim \boldsymbol{U} + \dim \boldsymbol{U}^* = n - 1.$$

Every projective transformation $f : \mathbb{RP}^n \to \mathbb{RP}^n \in \mathrm{PGL}(n)$ induces a *dual projective transformation* $f^* : (\mathbb{RP}^n)^* \to (\mathbb{RP}^n)^* \in \mathrm{PGL}(n)^*$ such that

$$f(\boldsymbol{U})^* = f^*(\boldsymbol{U}^*)$$

for every projective subspace $\boldsymbol{U} \subset \mathbb{RP}^n$. Introduce a basis on $\mathbb{R}^{n+1}$, say the conical basis, and its dual basis on $(\mathbb{R}^{n+1})^*$. Then, if $F \in \mathbb{R}^{n+1\times n+1}$ is a matrix representing the transformation $f = [F]$, a matrix $F^* \in \mathbb{R}^{n+1\times n+1}$ representing the dual transformation $f^* = [F^*]$ is given by

$$F^* := F^{-\intercal}. \tag{14}$$

For a quadric $\mathcal{Q} \subset \mathbb{RP}^n$ its *dual quadric* $\mathcal{Q}^* \subset (\mathbb{RP}^n)^*$ may be defined as the set of points dual to the tangent hyperplanes of $\mathcal{Q}$.

**Example A.1.**

(i) For a non-degenerate quadric $\mathcal{Q} \subset \mathbb{RP}^n$ of signature $(r, s)$ its dual quadric $\mathcal{Q}^* \subset (\mathbb{RP}^n)^*$ is non-degenerate with same signature.

(ii) For a cone $\mathcal{Q} \subset \mathbb{RP}^n$ of signature $(r, s, 1)$ with vertex $\boldsymbol{v} \in \mathcal{Q}$, its dual quadric $\mathcal{Q}^* \subset (\mathbb{RP}^n)^*$ consists of the set of points on a lower dimensional quadric of signature $(r, s)$ contained in the hyperplane $\boldsymbol{v}^* \subset (\mathbb{RP}^n)^*$.

## A.1 Euclidean geometry

Let $\langle\cdot,\cdot\rangle$ be the standard degenerate bilinear form of signature $(n, 0, 1)$, i.e.

$$\langle x, y\rangle := x_1y_1 + \ldots + x_ny_n$$

for $x, y \in \mathbb{R}^{n+1}$. The corresponding quadric $\mathcal{C}$ may be considered as an imaginary cone (cf. Example 1.1 (iv)), its real part only consisting of one point, the vertex of the cone,

$$\boldsymbol{m}_\infty \in \mathbb{RP}^n, \qquad m_\infty := e_{n+1} = (0, \ldots, 0, 1).$$

While the set $\mathcal{C}^- = \varnothing$ is empty, the set

$$\mathbf{E}^* := \mathcal{C}^+ = \mathbb{RP}^n \setminus \{\boldsymbol{m}_\infty\}$$

consists of the whole projective space but one point, which we identify with the $n$-dimensional *dual Euclidean space*, i.e., the space of Euclidean hyperplanes.

While in the projective models of hyperbolic/elliptic geometry, we were able to identify certain points with hyperplanes by polarity in the same projective space, this is not possible in projective model of Euclidean geometry due to the degeneracy of the absolute quadric $\mathcal{C}$. Instead, by duality, every point $\boldsymbol{m} \in \mathbf{E}^*$ corresponds to a hyperplane $\boldsymbol{m}^* \subset \mathbf{E}$ in

$$\mathbf{E} := (\mathbb{R}\mathrm{P}^n)^* \setminus (\boldsymbol{m}_\infty)^* \simeq \mathbb{R}^n,$$

which we identify with the $n$-dimensional *Euclidean space.* The hyperplane $(\boldsymbol{m}_\infty)^*$ is called the *hyperplane at infinity.*

For two points $\boldsymbol{k}, \boldsymbol{m} \in \mathbf{E}^*$ one always has $0 \leq K_{\mathcal{C}}(\boldsymbol{k}, \boldsymbol{m}) \leq 1$, and the Euclidean angle $\alpha$, or equivalently its conjugate angle $\pi - \alpha$, between the two hyperplanes $\boldsymbol{k}^*, \boldsymbol{m}^* \subset \mathbf{E}$ is given by

$$K_{\mathcal{C}}(\boldsymbol{k}, \boldsymbol{m}) = \cos^2 \alpha(\boldsymbol{k}^*, \boldsymbol{m}^*).$$

The two planes are *parallel* if the line $\boldsymbol{k} \wedge \boldsymbol{m}$ contains the point $\boldsymbol{m}_\infty$.

The dual object of the absolute cone $\mathcal{C}^*$ can be identified with an (imaginary) quadric in the hyperplane at infinity $(\boldsymbol{m}_\infty)^*$. Since this does not induce a bilinear form on $(\mathbb{R}\mathrm{P}^n)^*$, the Cayley-Klein distance on $\mathbf{E}$ is not well-defined. Yet the Euclidean distance may still be recovered in this setup, e.g., as the limit of the Cayley-Klein distance of hyperbolic/elliptic space [Kle1928, Gun2011]. One may avoid these difficulties by treating Euclidean geometry as a subgeometry of Möbius geometry (see Section A.2).

We may employ the following normalization for the dual Euclidean space

$$(\mathbb{E}^n)^* := \left\{ m \in \mathbb{R}^{n+1} \;\middle|\; \langle m, m \rangle = 1 \right\} = \left\{ (\widehat{m}, -d) \in \mathbb{R}^{n+1} \;\middle|\; \widehat{m} \in \mathbb{R}^n,\ d \in \mathbb{R},\ \widehat{m} \cdot \widehat{m} = 1 \right\}.$$

Upon the (non-invariant) identification $(\mathbb{R}^{n+1})^* \simeq \mathbb{R}^{n+1}$ by identifying the canonical basis of $(\mathbb{R}^{n+1})^*$ with the dual basis of the canonical basis of $\mathbb{R}^{n+1}$, we introduce the following normalization for the Euclidean space.

$$\mathbb{E}^n := \left\{ x \in (\mathbb{R}^{n+1})^* \;\middle|\; x(m_\infty) = 1 \right\} \simeq \left\{ (\widehat{x}, 1) \in \mathbb{R}^{n+1} \;\middle|\; \widehat{x} \in \mathbb{R}^n \right\}.$$

Then $\mathrm{P}(\mathbb{E}^n) = \mathbf{E}$ is an embedding and $\mathrm{P}((\mathbb{E}^n)^*) = \mathbf{E}^*$ a double cover. The double cover may be used to encode the orientation of the corresponding Euclidean plane.

In this normalization the Euclidean distance of two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{E}$, $x, y \in \mathbb{E}^n$ is given by

$$|x - y| = d(\boldsymbol{x}, \boldsymbol{y})$$

The Euclidean hyperplane corresponding to a point $\boldsymbol{m} \in \mathbf{E}^*$, $m = (\widehat{m}, -d) \in (\mathbb{E}^n)^*$ is given by

$$\{\boldsymbol{x} \in \mathbf{E} \mid m \cdot x = 0\} = \mathrm{P}\left(\{(\widehat{x}, 1) \in \mathbb{E}^n \mid \widehat{m} \cdot \widehat{x}\} = d\right),$$

while the formula for the angle between two Euclidean planes $\boldsymbol{k} \in \mathbf{E}^*$, $k = (\widehat{k}, -c) \in (\mathbb{E}^n)^*$ and $\boldsymbol{m} \in \mathbf{E}^*$, $m = (\widehat{m}, -d) \in (\mathbb{E}^n)^*$ becomes

$$\langle k, m \rangle = \widehat{k} \cdot \widehat{m} = \cos \alpha(\boldsymbol{k}^*, \boldsymbol{m}^*),$$

where the intersection angle and its conjugate angle can be distinguished now. Finally, the signed distance of a point $\boldsymbol{x} \in \mathbf{E}$, $x = (\widehat{x}, 1) \in \mathbb{E}^n$ and a plane $\boldsymbol{m} \in \mathbf{E}^*$, $(\widehat{m}, -d) \in (\mathbb{E}^n)^*$ is given by

$$\langle m, x \rangle = \widehat{m} \cdot \widehat{x} - d = d(\boldsymbol{x}, \boldsymbol{m}^*)$$

The transformation group induced by the absolute quadric $\mathcal{C}$ on the dual Euclidean space $\mathbf{E}^*$ is given by $\mathrm{PO}(n, 0, 1)$. Its elements are of the form

$$[A] = \left[ \begin{array}{c|c} \widehat{A} & 0 \\ \hline \widehat{a}^\intercal & \varepsilon \end{array} \right] \in \mathrm{PO}(n, 0, 1),$$

where $\widehat{A} \in O(n), \widehat{a} \in \mathbb{R}^n, \varepsilon \neq 0$. Thus, its dual transformations, see (14), are given by

$$[A^{-\intercal}] = \left[\begin{array}{c|c} \widehat{A} & -\widehat{A}\widehat{a} \\ \hline 0 & \varepsilon^{-1} \end{array}\right] \in \mathrm{PO}(n,0,1)^*.$$

They act on $\mathbf{E}$ as the group of *similarity transformations*, i.e., Euclidean motions and scalings.

## A.2 Euclidean geometry from Möbius geometry

In Section 3 we have excluded the choice of a point $\boldsymbol{q} \in \mathcal{Q}$ on the quadric, since the projection $\pi_{\boldsymbol{q}}$ (see Definition 3.1) to the polar hyperplane $\boldsymbol{q}^\perp$ is not well-defined in that case. Yet most of the constructions described still apply if we project to any other hyperplane instead. We show this in the example of recovering Euclidean (similarity) geometry from Möbius geometry.

Thus, let $\boldsymbol{q} \in \mathcal{S}$ be a point on the Möbius quadric, w.l.o.g.,

$$\boldsymbol{q} \coloneqq [e_\infty], \qquad e_\infty \coloneqq \tfrac{1}{2}\left(e_{n+1} + e_{n+2}\right) = (0, \ldots, 0, \tfrac{1}{2}, \tfrac{1}{2})$$

While $\boldsymbol{q}^\perp$ is the tangent plane of $\mathcal{S}$ at $\boldsymbol{q}$, we choose a different plane as the projection plane $\mathbf{B}$, w.l.o.g.,

$$\mathbf{B} \coloneqq [e_{n+1}]^\perp = [0, \ldots, 0, 1, 0]^\perp.$$

Consider the central projection from $\mathbf{B}$ to $\mathcal{S}$ through the point $\boldsymbol{q}$, which is also called *stereographic projection.*

To this end, denote by $[e_0]$ the intersection point of the line $[e_\infty] \wedge [k]$ with $\mathcal{S}$, where

$$e_0 \coloneqq \tfrac{1}{2}\left(e_{n+2} - e_{n+1}\right) = (0, \ldots, 0, -\tfrac{1}{2}, \tfrac{1}{2}).$$

Then we have

$$\langle e_0, e_0 \rangle = \langle e_\infty, e_\infty \rangle = 0, \quad \langle e_0, e_\infty \rangle = -\tfrac{1}{2},$$

and $\langle e_0, e_i \rangle = \langle e_\infty, e_i \rangle = 0$ for $i = 1, \ldots, n$, and the vectors $e_1, \ldots, e_n, e_0, e_\infty$ build a basis of $\mathbb{R}^{n+1,1}$.

**Proposition A.1.** *Let $\ell \coloneqq \mathbf{B} \cap \boldsymbol{q}^\perp$. The stereographic projection from $\mathbf{B} \setminus \ell$ to $\mathcal{S} \setminus \boldsymbol{q}$ through the point $\boldsymbol{q}$ is given by the map*

$$\sigma_{\boldsymbol{q},\boldsymbol{k}} : \boldsymbol{x} = [\tilde{x} + e_0 - e_\infty] \mapsto [\tilde{x} + e_0 + |\tilde{x}|^2 \, e_\infty],$$

*where $\tilde{x} \in \operatorname{span}\{e_1, \ldots, e_n\}$.*

*Proof.* First note that a point in $\boldsymbol{x} \in \mathbf{B} \setminus \ell$ may be normalized to $x = \tilde{x} + e_0 - e_\infty$ The (second) intersection point of the line $\boldsymbol{q} \wedge \boldsymbol{x}$ with $\mathcal{Q}$ is then given by

$$-2\left\langle x, e_\infty \right\rangle x + \left\langle x, x \right\rangle e_\infty = x + (|\tilde{x}|^2 - 1)e_\infty = \tilde{x} + e_0 + |\tilde{x}|^2 \, e_\infty.$$

□

Now the Euclidean metric on $\mathbf{B}$ may be recovered from the bilinear form corresponding to $\mathcal{S}$ by observing that

$$\langle x, y \rangle = \left\langle \tilde{x} + e_0 + |\tilde{y}|^2 \, e_\infty, \tilde{y} + e_0 + |\tilde{y}|^2 \, e_\infty \right\rangle = -\frac{1}{2} \left| \tilde{x} - \tilde{y} \right|.$$

*Remark* A.1. To obtain the Euclidean metric in a projectively well-defined way one can start by considering the quantity

$$\frac{\langle x, y \rangle}{\langle e_\infty, x \rangle \, \langle e_\infty, y \rangle},$$

similar to Definition B.1. Though not being invariant under different choices of homogeneous coordinate vectors for the point $\boldsymbol{q} = [e_\infty]$, the quotient of two such expressions is. This fits the fact that it not actually Euclidean geometry that we are recovering but similarity geometry.

The restriction of the Möbius quadric $\mathcal{S}$ to the tangent hyperplane $\boldsymbol{q}^\perp$ yields a quadric of signature $(n, 0, 1)$. Thus, we can identify the tangent hyperplane with the dual Euclidean space (see Section A.1). Indeed, by polarity in the Möbius quadric $\mathcal{S}$ every point $\boldsymbol{k} \in \boldsymbol{q}^\perp$ corresponds to hyperplanar section of $\mathcal{S}$ containing the point $\boldsymbol{q}$, i.e. a $\mathcal{S}$-sphere through $\boldsymbol{q}$, which is, in turn, mapped to a hyperplane of $\mathbf{B}$ by stereographic projection. The Cayley-Klein distance of two points in the tangent hyperplane yields the Euclidean angle between the two corresponding hyperplanes of $\mathbf{B}$. The group of Möbius transformations fixing the point $\boldsymbol{q}$ induces the group of dual similarity transformations on $\mathbf{B}$

$$\mathbf{Mob}_{\boldsymbol{q}} = \mathrm{PO}(n+1, 1)_{\boldsymbol{q}} \simeq \mathrm{PO}(n, 0, 1).$$

## A.3 Euclidean Laguerre geometry

In the spirit of Section 3 the absolute quadric $\mathcal{C} \subset \mathbb{RP}^n$ of the dual Euclidean (similarity) space with signature $(n, 0, 1)$ can be lifted to a quadric $\mathcal{B}_{\mathrm{euc}} \subset \mathbb{RP}^{n+1}$ of signature $(n, 1, 1)$, we call the *Euclidean Laguerre quadric*, or classically the *Blaschke cylinder*. The group of *Euclidean Laguerre transformations* is given by

$$\mathbf{Lag}_{\mathrm{euc}} = \mathrm{PO}(n, 1, 1).$$

Now for a point $\boldsymbol{p}$ with $\langle p, p \rangle < 1$, w.l.o.g.,

$$\boldsymbol{p} \coloneqq [0, \cdots, 0, 1, 0]$$

the involution $\sigma_{\boldsymbol{p}}$ and projection $\pi_{\boldsymbol{p}}$ (see Definition 3.1) are still well-defined, and the quotient

$$(\mathbf{Lag}_{\mathrm{euc}})_{\boldsymbol{p}} / \sigma_{\boldsymbol{p}} \simeq \mathrm{PO}(n, 0, 1)$$

recovers the group of dual Euclidean (similarity) transformations.

The projection $\pi_{\boldsymbol{p}}$ restricted to $\mathcal{B}_{\mathrm{euc}}$ realizes a double cover of the dual Euclidean space $\mathcal{C}^+ = \mathbf{E}^*$, which can be interpreted as carrying the information of the orientation of the corresponding hyperplanes in $\mathbf{E}$. The involution $\sigma_{\boldsymbol{p}}$ plays again the role of orientation reversion.

The hyperplanar sections of $\mathcal{B}_{\mathrm{euc}} \subset \mathbb{RP}^{n+1}$ correspond to (the tangent hyperplanes) of a Euclidean sphere in $\mathbf{E}$. Yet due to the degeneracy of $\mathcal{B}_{\mathrm{euc}}$ they cannot be identified with (polar) points in the same space. Instead they can be identified with points in the dual space $(\mathbb{RP}^{n+1})^*$, which is classically called the *cyclographic model* of Laguerre geometry. The dual quadric $\mathcal{B}^*_{\mathrm{euc}}$ is given by a lower dimensional quadric of signature $(n, 1)$ contained in the hyperplane $\boldsymbol{m}^*_\infty$. Thus, the cyclographic model is isomorphic to $(n+1)$-dimensional *Minkowski space*.

## A.4 Lie geometry in Euclidean space

A Euclidean model of Lie geometry is obtained by stereographic projection of the point complex $\mathcal{S} \subset \mathcal{L}$.

We write the bilinear form corresponding to the Lie quadric as

$$\langle x, y \rangle \coloneqq \widehat{x} \cdot \widehat{y} - x_{n+2} y_{n+2} - x_{n+3} y_{n+3} \;=\; \sum_{i=1}^{n+1} x_i y_i - x_{n+2} y_{n+2} - x_{n+3} y_{n+3}$$

for $x, y \in \mathbb{R}^{n+3}$, where

$$\widehat{\cdot} : \mathbb{R}^{n+3} \to \mathbb{R}^{n+1}, \quad (x_1, \ldots, x_{n+3}) \mapsto (x_1, \ldots, x_{n+1}).$$

The point complex $\mathcal{S}$ is projectively equivalent to $\mathbb{S}^n$. Indeed, $\boldsymbol{p}^\perp = \{\boldsymbol{x} \in \mathbb{RP}^{n+2} \mid x_{n+3} = 0\} \simeq \mathbb{RP}^{n+1}$, and for a point $\boldsymbol{x} = [\widehat{x}, 1, 0] \in \boldsymbol{p}^\perp$ we find that in affine coordinates $(x_{n+2} = 1)$

$$\langle x, x \rangle = 0 \;\Leftrightarrow\; \widehat{x} \cdot \widehat{x} = 1.$$

Thus, we obtain the identification

$$\mathcal{S} = \left\{ \boldsymbol{x} \in \boldsymbol{p}^{\perp} \;\middle|\; \langle x, x \rangle = 0 \right\} \simeq \left\{ \widehat{x} \in \mathbb{R}^{n+1} \;\middle|\; \widehat{x} \cdot \widehat{x} = 1 \right\} = \mathbb{S}^n.$$

We embed the sphere $\mathbb{S}^n$ into the light cone

$$\mathbb{L}^{n+1,2} = \left\{ x \in \mathbb{R}^{n+3} \;\middle|\; \langle x, x \rangle = 0 \right\}$$

in the following way

$$\sigma_{\mathbb{S}^N} : \mathbb{S}^n \hookrightarrow \mathbb{L}^{n+1,2}, \quad \widehat{x} \mapsto \; \widehat{x} + e_{n+2} + 0 \cdot e_{n+3}.$$

Then we have $\mathcal{S} = \mathrm{P}(\sigma_{\mathbb{S}^N}(\mathbb{S}^n))$, where P acts one-to-one on the image of $\sigma_{\mathbb{S}^N}$.

Denote

$$e_\infty := \tfrac{1}{2}\left(e_{n+2} + e_{n+1}\right), \quad e_0 := \tfrac{1}{2}\left(e_{n+2} - e_{n+1}\right),$$

which are homogeneous coordinate vectors for the *north pole* and *south pole* of $\mathcal{S} \simeq \mathbb{S}^n$ respectively. They satisfy

$$\langle e_0, e_0 \rangle = \langle e_\infty, e_\infty \rangle = 0, \quad \langle e_0, e_\infty \rangle = -\tfrac{1}{2},$$

and $\langle e_0, e_i \rangle = \langle e_\infty, e_i \rangle = 0$ for $i = 1, \ldots, n, n+3$. The vectors $e_1, \ldots, e_n, e_0, e_\infty, e_{n+3}$ build a new basis of $\mathbb{R}^{n+1,2}$. We define an embedding of $\mathbb{R}^n$ into the light cone $\mathbb{L}^{n+1,2}$ by the map

$$\sigma_{\mathbb{R}^N} : \mathbb{R}^n \hookrightarrow \mathbb{L}^{n+1,2}, \quad \tilde{x} \mapsto \; \tilde{x} + e_0 + |\tilde{x}|^2 e_\infty + 0 \cdot e_{n+3}$$

and recognize that upon renormalizing the $(n+2)$-nd coordinate to 1 this is nothing but stereographic projection from $\mathbb{R}^n$ onto the sphere $\mathbb{S}^n$, i.e.

$$(\sigma_{\mathbb{S}^N})^{-1} \circ \sigma_{\mathbb{R}^N} : \mathbb{R}^n \to \mathbb{S}^n, \quad \tilde{x} \mapsto \left( \frac{2\tilde{x}}{|\tilde{x}|^2 + 1}, \frac{1 - |\tilde{x}|^2}{1 + |\tilde{x}|^2} \right),$$

and $\mathcal{S} = \; P(\sigma_{\mathbb{S}^N}(\mathbb{S}^n)) = \mathrm{P}(\sigma_{\mathbb{R}^N}(\mathbb{R}^n)) \cup \{[e_\infty]\}$. Every point $\boldsymbol{s} \in \mathcal{L}$ with $s_0 \neq 0$ can be represented by

$$s = \tilde{s} + e_0 + (|\tilde{s}|^2 - r^2)e_\infty + r e_{n+3}$$

with $\tilde{s} \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Then for $x = \tilde{x} + e_0 + |\tilde{x}|^2 e_\infty$ we find

$$\langle s, x \rangle = 0 \quad \Leftrightarrow \quad |\tilde{s} - \tilde{x}|^2 = r^2.$$

Thus we may identify the point $\boldsymbol{s}$ with the oriented Euclidean hypersphere of $\mathbb{R}^n$ with center $\tilde{s}$ and signed radius $r \in \mathbb{R}$. Analogously a point $\boldsymbol{n} \in \mathcal{L}$ with $n_0 = 0$ may be represented by

$$n = \tilde{n} + 0 \cdot e_0 + 2d e_\infty + e_{n+3}$$

and identified with the oriented hyperplane of $\mathbb{R}^n$ with normal $\tilde{n} \in \mathbb{S}^{n-1}$ and signed distance $d \in \mathbb{R}$ to the origin.

**Proposition A.2.** *Under the aforementioned identification two oriented hyperspheres/hyperplanes of Euclidean space are in oriented contact if and only if the their corresponding points on the Lie quadric are Lie orthogonal.*

*Proof.* For, e.g., two oriented hyperspheres of $\mathbb{R}^n$ represented by homogeneous coordinate vectors $s_i = \tilde{s}_i + e_0 + (|\tilde{s}_i|^2 - r_i^2)e_\infty + r_i e_{n+3}$, $i = 1, 2$ we find

$$\langle s_1, s_2 \rangle = 0 \;\Leftrightarrow\; |\tilde{s}_1 - \tilde{s}_2|^2 = (r_1 - r_2)^2.$$

□

| **Euclidean geometry** | **Lie geometry** |
|---|---|
| *point* $\tilde{x} \in \mathbb{R}^n$ | $[\tilde{x} + e_0 + \lvert x\rvert^2 e_\infty + 0 \cdot e_{n+3}] = \left[\tilde{x}, \frac{1-\lvert\tilde{x}\rvert^2}{2}, \frac{1+\lvert\tilde{x}\rvert^2}{2}, 0\right] \in \mathcal{L}$ |
| *oriented hypersphere* with center $\tilde{s} \in \mathbb{R}^n$ and signed radius $r \in \mathbb{R}$ | $[\tilde{s} + e_0 + (\lvert\tilde{s}\rvert^2 - r^2)e_\infty + re_{n+3}] = \left[\tilde{s}, \frac{1-\lvert\tilde{s}\rvert^2+r^2}{2}, \frac{1+\lvert\tilde{s}\rvert^2-r^2}{2}, r\right] \in \mathcal{L}$ |
| *oriented hyperplane* $\langle\tilde{n}, \tilde{x}\rangle = d$, with normal $\tilde{n} \in \mathbb{S}^{n-1}$ and signed distance $d \in \mathbb{R}$ | $[\tilde{n} + 0 \cdot e_0 + 2de_\infty + e_{n+3}] = [\tilde{n}, -2d, 2d, 1] \in \mathcal{L}$ |

**Table 7.** Correspondence between the geometric objects of Lie geometry in Euclidean space and points on the Lie quadric.

The condition $n_0 = 0$, which characterizes the oriented hyperplanes among all oriented hyperspheres, is equivalent to $\langle n, e_\infty\rangle = 0$. Thus, we can interpret oriented hyperplanes as oriented hyperspheres containing the point $\boldsymbol{q} := [e_\infty]$. Similar to the point complex (see Definition 5.3), we may introduce the *Euclidean plane complex* (cf. Definition 5.5)

$$\mathcal{L} \cap \boldsymbol{q}^\perp \simeq \mathcal{B}_{\text{euc}} \tag{15}$$

representing all oriented hyperplanes of $\mathbb{R}^n$. The Euclidean plane complex is a parabolic sphere complex (see Definition 5.4). Its signature is given by $(n, 1, 1)$, and we recover Euclidean Laguerre geometry (cf. Section A.3) by considering the action on $\boldsymbol{q}^\perp$ of all Lie transformations that fix the point $\boldsymbol{q}$

$$\mathbf{Lie}_{\boldsymbol{q}} \simeq \mathbf{Lag}_{\text{euc}}.$$

# B Invariant on a quadric induced by a point

While two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{Q}$ on a quadric $\mathcal{Q} \subset \mathbb{RP}^{n+1}$ with $\langle x, y\rangle \neq 0$ possess no projective invariant, the additional choice of the point $\boldsymbol{q}$ allows for the definition of the following "distance".

**Definition B.1.** Let $\boldsymbol{q} \in \mathbb{RP}^{n+1} \setminus \mathcal{Q}$. Then we call

$$I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{y}) \coloneqq 1 - \frac{\langle x, y\rangle \, \langle q, q\rangle}{\langle x, q\rangle \, \langle y, q\rangle}.$$

the $\boldsymbol{q}$-*distance* for any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{Q}$.

*Remark* B.1. Although we are interested in the $\boldsymbol{q}$-distance for points on the quadric for now, it can be extended to all of $\mathbb{RP}^{n+1} \setminus \boldsymbol{q}^\perp$. Then the relation between the $\boldsymbol{q}$-distance and the Cayley-Klein distance induced by $\mathcal{Q}$ is given by

$$K_{\mathcal{Q}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{(1 - I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{y}))^2}{(1 - I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{x}))(1 - I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{y}, \boldsymbol{y}))}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{RP}^{n+1} \setminus (\mathcal{Q} \cup \boldsymbol{q}^\perp)$.

The $\boldsymbol{q}$-distance is projectively well-defined, in the sense that it does not depend on the choice of homogeneous coordinate vectors for the points $\boldsymbol{q}$, $\boldsymbol{x}$, and $\boldsymbol{y}$, and it is invariant under the action of the group $\mathrm{PO}(r, s, t)_{\boldsymbol{q}}$:

**Proposition B.1.** *Let $\boldsymbol{q} \in \mathbb{RP}^{n+1} \setminus \mathcal{Q}$. Then the $\boldsymbol{q}$-distance is invariant under all projective transformations that preserve the quadric $\mathcal{Q}$ and fix the point $\boldsymbol{q}$, i.e.*

$$I_{\mathcal{Q},\boldsymbol{q}}(F(\boldsymbol{x}), F(\boldsymbol{y})) = I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{x})$$

*for $F \in \mathrm{PO}(r, s, t)_{\boldsymbol{q}}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{Q}$.*

Applying the involution $\sigma_{\boldsymbol{q}}$ to only one of the arguments of the $\boldsymbol{q}$-distance results in a change of sign.

**Proposition B.2.** *Let $\boldsymbol{q} \in \mathbb{RP}^{n+1} \setminus \mathcal{Q}$. Then the $\boldsymbol{q}$-distance satisfies*

$$I_{\mathcal{Q},\boldsymbol{q}}(\sigma_{\boldsymbol{q}}(\boldsymbol{x}), \boldsymbol{y}) = I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \sigma_{\boldsymbol{q}}(\boldsymbol{y})) = -I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{y}).$$

*for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{Q}$.*

*Proof.* Using Definitions B.1 and 3.1 we obtain

$$\begin{aligned} I_{\mathcal{Q},\boldsymbol{q}}(\sigma_{\boldsymbol{q}}(\boldsymbol{x}), \boldsymbol{y}) &= 1 - \frac{\langle \sigma_q(x), y\rangle \, \langle q, q\rangle}{\langle \sigma_q(x), q\rangle \, \langle y, q\rangle} = 1 - \frac{\langle x, y\rangle \, \langle q, q\rangle - 2\langle x, q\rangle \, \langle y, q\rangle}{-\langle x, q\rangle \, \langle y, q\rangle} \\ &= \frac{\langle x, y\rangle \, \langle q, q\rangle}{\langle x, q\rangle \, \langle y, q\rangle} - 1 = -I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{y}). \end{aligned}$$

□

Thus, we find that the square of the $\boldsymbol{q}$-distance is well-defined on the quotient ${}^{\mathcal{Q}}/_{\sigma_{\boldsymbol{q}}}$, which, according to Proposition 3.1, can be identified with its projection $\pi_{\boldsymbol{q}}(\mathcal{Q})$ to the plane $\boldsymbol{q}^\perp$. In this projection the square of the $\boldsymbol{q}$-distance becomes the Cayley-Klein distance induced by $\widetilde{\mathcal{Q}} = \mathcal{Q} \cap \boldsymbol{q}^\perp$ (see Proposition 3.3)

$$I_{\mathcal{Q},\boldsymbol{q}}(\boldsymbol{x}, \boldsymbol{y})^2 = K_{\widetilde{\mathcal{Q}}}(\pi_{\boldsymbol{q}}(\boldsymbol{x}), \pi_{\boldsymbol{q}}(\boldsymbol{y})).$$

Hypersurfaces of $\mathcal{Q}$ of constant $\boldsymbol{q}$-distance to point on $\mathcal{Q}$ are hyperplanar sections of $\mathcal{Q}$, i.e. the $\mathcal{Q}$-spheres (see Definition 3.2).

**Proposition B.3.** *The hypersurface in $\mathcal{Q}$ of constant $\boldsymbol{q}$-distance $\nu \in \mathbb{R}$ to a point $\tilde{\boldsymbol{x}} \in \mathcal{Q}$ is given by the intersection with the polar hyperplane of the point $\boldsymbol{x} \in \mathbb{RP}^{n+1}$,*

$$x := \langle q, q \rangle \; \tilde{x} + (\nu - 1) \langle \tilde{x}, q \rangle \; q,$$

*i.e.*

$$\{\boldsymbol{y} \in \mathcal{Q} \mid I_{\mathcal{Q},\boldsymbol{q}} (\tilde{\boldsymbol{x}}, \boldsymbol{y}) = \nu\} \;\; = \;\; \boldsymbol{x}^\perp \cap \mathcal{Q}.$$

*Proof.* The equation

$$I_{\mathcal{Q},\boldsymbol{q}} (\tilde{\boldsymbol{x}}, \boldsymbol{y}) = 1 - \frac{\langle \tilde{x}, y \rangle \; \langle q, q \rangle}{\langle \tilde{x}, q \rangle \; \langle y, q \rangle} = \nu$$

is equivalent to

$$\langle x, y \rangle = \langle q, q \rangle \; \langle \tilde{x}, y \rangle + (\nu - 1) \langle \tilde{x}, q \rangle \; \langle q, y \rangle = 0.$$

□

But are all hyperplanar sections of $\mathcal{Q}$ such hypersurfaces (cf. Proposition 3.4)? Following Proposition B.3 the potential centers of a given planar section $\boldsymbol{x}^\perp \cap \mathcal{Q}$ are given by the intersection points of the line $\boldsymbol{q} \wedge \boldsymbol{x}$ with the quadric $\mathcal{Q}$. Yet such lines do not always intersect the quadric in real points.

**Proposition B.4.** *Denote by*

$$\Delta_q(x) := \langle x, q \rangle^2 - \langle x, x \rangle \; \langle q, q \rangle = - \langle q, q \rangle \; \langle x, x \rangle_q$$

*the quadratic form of the cone of contact $C_{\mathcal{Q}}(\boldsymbol{q})$. Let $\boldsymbol{x} \in \mathbb{RP}^{n+1}$ such that $\boldsymbol{x}^\perp \cap \mathcal{Q} \neq \varnothing$.*

- *If $\Delta_q(x) > 0$, then the line $\boldsymbol{q} \wedge \boldsymbol{x}$ intersects the quadric $\mathcal{Q}$ in two (real) points, and*

  $$\boldsymbol{x}^\perp \cap \mathcal{Q} = \{\boldsymbol{y} \in \mathcal{Q} \mid I_{\mathcal{Q},\boldsymbol{q}} (\boldsymbol{x}_\pm, \boldsymbol{y}) = \nu_\pm\}$$

  *with*

  $$x_\pm = \langle q, q \rangle \, x + \left( - \langle x, q \rangle \pm \sqrt{\Delta} \right) q, \qquad \nu_\pm := \pm \frac{\langle x, q \rangle}{\sqrt{\Delta}}.$$

- *If $\Delta_q(x) < 0$, then the line $\boldsymbol{q} \wedge \boldsymbol{x}$ intersects the quadric $\mathcal{Q}$ in two complex conjugate points, and*

  $$\boldsymbol{x}^\perp \cap \mathcal{Q} = \{\boldsymbol{y} \in \mathcal{Q} \mid I_{\mathcal{Q},\boldsymbol{q}} (\boldsymbol{x}_\pm, \boldsymbol{y}) = \nu_\pm\}$$

  *with*

  $$x_\pm = \langle q, q \rangle \, x + \left( - \langle x, q \rangle \pm i\sqrt{-\Delta} \right) q, \qquad \nu_\pm := \pm \frac{\langle x, q \rangle}{i\sqrt{-\Delta}}.$$

*Proof.* The first equality for the quadratic form of the cone of contact follows from Lemma 1.7, while the second equality immediately follows from substituting $x = \alpha q + \pi_q(x)$.

In the case $\Delta_q(x) \neq 0$ the form of the intersection points $\boldsymbol{x}_\pm$ follows from Lemma 1.6. Substituting into the $\boldsymbol{q}$-distance gives, e.g., in the case $\Delta_q(x) > 0$

$$I_{\mathcal{Q},\boldsymbol{q}} (\boldsymbol{x}_\pm, \boldsymbol{y}) = 1 - \frac{(- \langle x, q \rangle \pm \sqrt{\Delta}) \, \langle q, q \rangle}{\langle x_\pm, q \rangle} = \pm \frac{\langle x, q \rangle}{\sqrt{\Delta}},$$

where we used $\langle x, q \rangle = \pm \sqrt{\Delta} \, \langle q, q \rangle$. □

We denoted the space of $\mathcal{Q}$-spheres by $\mathfrak{S}$ (see Definition 3.2).

*Remark* B.2. Two points $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}} \in \mathfrak{S}$ describing two $\boldsymbol{q}$-spheres with centers $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{Q}$ and $\boldsymbol{q}$-radii $\nu_1, \nu_2$ have $\boldsymbol{q}$-distance

$$I_{\mathcal{Q},\boldsymbol{q}} (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \frac{I_{\mathcal{Q},\boldsymbol{q}} (\boldsymbol{x}, \boldsymbol{y})}{\nu_1 \nu_2}.$$

Note that change of the representing center and radius, e.g. $\boldsymbol{x} \to \sigma_{\boldsymbol{q}}(\boldsymbol{x})$, $\nu_1 \to -\nu_1$, leaves the resulting quantity invariant.

## B.1 Signed inversive distance

We first give a Euclidean definition for the signed inversive distance.

**Definition B.2.** The *signed inversive distance* of two oriented hyperspheres in $\mathbb{R}^n$ with centers $\tilde{s}_1, \tilde{s}_2 \in \mathbb{R}^n$ and signed radii $r_1, r_2 \in \mathbb{R}$ is given by

$$I := \frac{r_1^2 + r_2^2 - |\tilde{s}_1 - \tilde{s}_2|^2}{2r_1r_2}.$$

In particular, if the two spheres intersect it is the cosine of their intersection angle, by the cosine law for Euclidean triangles.

This classical invariant is usually given in its unsigned version, which was introduced by Coxeter [Cox1971] as a Möbius invariant.

**Proposition B.5.** *The signed inversive distance $I$ satisfies*

- $I \in (-1, 1) \Leftrightarrow$ *the two oriented hyperspheres intersect. In this case $I = \cos\varphi$ where $\varphi \in [0, \pi]$ is the angle between the two oriented hyperspheres.*
- $I = 1 \Leftrightarrow$ *the two oriented hyperspheres touch with matching orientation (Lie incidence).*
- $I = -1 \Leftrightarrow$ *the two oriented hyperspheres touch with opposite orientation.*
- $I \in (\infty, -1) \cup (1, \infty) \Leftrightarrow$ *the two oriented hyperspheres are disjoint.*

The signed inversive distance is nothing but the $\boldsymbol{p}$-distance (see Definition B.1) associated with the point complex $\mathcal{S} \subset \mathcal{L}$ in Lie geometry (see Definition 5.3), where

$$\boldsymbol{p} = [0, \cdots, 0, 1] \in \mathbb{RP}^{n+2}.$$

**Proposition B.6.** *For two oriented hyperspheres represented by*

$$\boldsymbol{s}_i = [\tilde{s}_i + e_0 + (|\tilde{s}_i|^2 - r_i^2)e_\infty + r_i e_{n+3}], \qquad i = 1, 2$$

*with Euclidean centers $\tilde{s}_1, \tilde{s}_2 \in \mathbb{R}^n$ and signed radii $r_1, r_2 \neq 0$ the $\boldsymbol{p}$-distance associated with the point complex $\mathcal{S}$ is equal to the signed inversive distance, i.e.*

$$I_{\mathcal{L},\boldsymbol{p}}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{r_1^2 + r_2^2 - |\tilde{s}_1 - \tilde{s}_2|^2}{2r_1r_2}.$$

*Proof.* With the given representation of the hyperspheres we find

$$I_{\mathcal{L},\boldsymbol{p}}(\boldsymbol{s}_1, \boldsymbol{s}_2) = 1 - \frac{\langle s_1, s_2\rangle \langle p, p\rangle}{\langle s_1, p\rangle \langle s_2, p\rangle} = 1 + \frac{(r_1^2 + r_2^2 - 2r_1r_2) - |\tilde{s}_1 - \tilde{s}_2|^2}{2r_1r_2} = \frac{r_1^2 + r_2^2 - |\tilde{s}_1 - \tilde{s}_2|^2}{2r_1r_2}.$$

□

Since we have expressed the signed inversive distance in terms of the $\boldsymbol{p}$-distance it follows that it is similarly well-defined for two oriented hyperspheres of $\mathbb{S}^n$. Furthermore, the signed inversive distance is invariant under all Lie transformations that preserve the point complex $\mathcal{S}$, i.e. all Möbius transformations.

As follows from Proposition 3.3 the Cayley-Klein distance of Möbius geometry, i.e. the Cayley-Klein distance induced by $\mathcal{S}$ onto $\boldsymbol{p}^\perp$ is the squared inversive distance.

**Proposition B.7.** *The Cayley-Klein distance of two points $[s_1], [s_2] \in \mathcal{S}^+$ is equal to their inversive distance.*

In particular, the intersection angle of spheres is a Möbius invariant.

## B.2 Geometric interpretation for sphere complexes

We now use the inversive distance to give a geometric interpretation for most sphere complexes in Lie geometry (see Definition 5.4). Let again

$$\boldsymbol{p} = [0, \cdots, 0, 1] \in \mathbb{R}\mathrm{P}^{n+2},$$

distinguishing the point complex $\mathcal{S} = \mathcal{L} \cap \boldsymbol{p}^{\perp}$.

**Proposition B.8.** *Let $\boldsymbol{q} \in \mathbb{R}\mathrm{P}^{n+2}$, $\boldsymbol{q} \neq \boldsymbol{p}$ such that the line $\boldsymbol{p} \wedge \boldsymbol{q}$ through $\boldsymbol{p}$ and $\boldsymbol{q}$ intersects the Lie quadric in two points, i.e. $\boldsymbol{p} \wedge \boldsymbol{q}$ has signature $(+-)$. Denote by*

$$\{\boldsymbol{q}_+, \boldsymbol{q}_-\} \coloneqq (\boldsymbol{p} \wedge \boldsymbol{q}) \cap \mathcal{L}$$

*the two intersection points of this line with the Lie quadric (the oriented hyperspheres corresponding to $\boldsymbol{q}_+$ and $\boldsymbol{q}_-$ only differ by their orientation).*

*Then the sphere complex corresponding to the point $\boldsymbol{q}$ is given by the set of oriented hyperspheres that have some fixed constant inversive distance $I_{\mathcal{L},\boldsymbol{p}}$ to the oriented hypersphere corresponding to $\boldsymbol{q}_+$, or equivalently, fixed constant inversive distance $-I_{\mathcal{L},\boldsymbol{p}}$ to the oriented hypersphere corresponding to $\boldsymbol{q}_-$.*

*In particular, in this case the sphere complex is*

- ***elliptic** if $I_{\mathcal{L},\boldsymbol{p}} \in (-1, 1)$,*
- ***hyperbolic** if $I_{\mathcal{L},\boldsymbol{p}} \in (-\infty, -1) \cup (1, \infty)$,*
- ***parabolic** if $I_{\mathcal{L},\boldsymbol{p}} \in \{-1, 1\}$.*

*Proof.* The two points $\boldsymbol{q}_\pm$ may be represented by

$$q_\pm = \tilde{q} + e_0 + \left(|\tilde{q}|^2 - R^2\right) e_\infty \pm R e_{n+3},$$

with some $R \neq 0$, where we assumed that the $e_0$-component of $q$ does not vanish. The case with $\langle q, e_\infty \rangle = 0$, which corresponds to $\boldsymbol{q}_\pm$ being planes, may be treated analogously.

Now the point $\boldsymbol{q}$ may be represented by

$$q = \tilde{q} + e_0 + \left(|\tilde{q}|^2 - R^2\right) e_\infty + \kappa e_{n+3}$$

with some $\kappa \in \mathbb{R}$. For any point $\boldsymbol{s} \in \mathcal{L}$ represented by

$$s = \tilde{q} + e_0 + \left(|\tilde{q}|^2 - r^2\right) e_\infty + r e_{n+3},$$

we find that the condition to lie on the sphere complex is given by

$$\langle q, s \rangle = 0 \quad \Leftrightarrow \quad \langle q, s \rangle_p = r\kappa.$$

Thus, the signed inversive distance of $\boldsymbol{q}_+$ and $\boldsymbol{s}$ is given by

$$I_{\boldsymbol{p}}(\boldsymbol{q}_+, \boldsymbol{s}) = 1 - \frac{\langle q_+, s \rangle \langle p, p \rangle}{\langle q_+, p \rangle \langle s, p \rangle} = \frac{\langle s, q \rangle_p}{rR} = \frac{\kappa}{R}$$

The change $\boldsymbol{q}_+ \to \boldsymbol{q}_-$ is equivalent to $R \to -R$ which leads to $I \to -I$.

The distinction of the three types of sphere complexes in terms of the value of the inversive distance is obtained by observing that

$$\begin{aligned}
\langle q, q \rangle &> 0, \quad \text{if } \kappa^2 < R^2, \\
\langle q, q \rangle &< 0, \quad \text{if } \kappa^2 > R^2, \\
\langle q, q \rangle &= 0, \quad \text{if } \kappa^2 = R^2.
\end{aligned}$$

$\square$

*Remark* B.3. For an elliptic sphere complex the line $\boldsymbol{p} \wedge \boldsymbol{q}$ always has signature $(+-)$. Furthermore, in this case we have $I_{\boldsymbol{p}} \in (-1, 1)$. Thus, according to Proposition B.5, any elliptic sphere complex is given by all oriented hyperspheres with constant angle to some fixed oriented hypersphere.

For hyperbolic sphere complexes the line $\boldsymbol{p} \wedge \boldsymbol{q}$ can have signature $(+-)$, $(--)$, or $(-0)$. The first case is captured by Proposition B.8. An example for signature $(--)$ is given by $\boldsymbol{q} = [0, \sin R, \cos R]$, which describes all oriented hyperspheres of $\mathbb{S}^n$ with spherical radius $R$. An example for signature $(-0)$ is given by $\boldsymbol{q} = [-2Re_\infty + e_{n+3}]$, which describes all oriented hyperspheres of $\mathbb{R}^n$ with (Euclidean) radius $R$. Note that the point complex $\mathcal{S}$ itself is also a hyperbolic sphere complex.

Parabolic sphere complexes are captured by Proposition B.8 if and only if $\boldsymbol{q} \notin \mathcal{S}$. Note that the (Euclidean) plane complex (15) is parabolic.

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