Abstracts

Niklas Affolter (TU Vienna)

Discrete maximal Lorentz surfaces and s-embeddings

S-embeddings were introduced by Chelkak as a generalization of Smirnov's approach to study the Ising model. We build upon the work of Chelkak, Laslier and Russkikh to present a class of s-embeddings that corresponds to discrete isothermic surfaces in Lorentz space. As a special case, we identify discrete maximal surfaces, which are discrete surfaces with vanishing discrete mean curvature. In this way, we introduce a result on the discrete level that was obtained by CLR in the limit. We also introduce an associated family of discrete maximal surfaces and the corresponding family of s-embeddings.
Joint work with Dellinger, Müller, Polly, Smeenk and Techter.

Nikolai Bobenko (University of Geneva)

Dimers and M-Curves: Limit Shapes from Riemann Surfaces

We present a general scheme for the study of dimer models via integrable systems techniques. This results in dimer models with quasi periodic weights. Putting an M-curve at the center of the construction allows one to define weights and algebraic objects describing the behavior of the corresponding dimer model. We obtain explicit formulas for the limit shapes of these models for certain boundary conditions. Furthermore everything in this approach can be computed numerically and the results match simulation.
This talk is based on joint work with Alexander I. Bobenko and Yuri B. Suris.

Gaëtan Borot (HU Berlin)

Macroscopic asymptotics in 2d random tilings

In the two-dimensional triangular lattice one can form three types of rhombi by joining two neighborhing triangles. Take a polygonal domain and choose a way (uniformly at random) to fill it with rhombi. Then dilate the domain by a factor L: one is interested in understanding the distribution of rhombi as L becomes large. There is a law of large numbers ("arctic circle phenomenon") and it is expected (Nekrasov, Okounkov) that the macroscopic fluctuations are described by a Gaussian free field. The type of rhombi meeting a vertical section of the domain is encoded into a random ensemble of particles along a line, behaving similarly to the eigenvalues of a random matrix but in a discrete setting. I will explain how combining large deviation techniques, the analysis of discrete Dyson–Schwinger equations, and the study of the geometry of spectral curves can help understanding the macroscopic behavior of this particle system, in particular for random tilings in non-simply connected domain.
This is based on joint ongoing work with Vadim Gorin and Alice Guionnet.

Cédric Boutiller (LPSM – Sorbonne Université)

Integrable Laplacians on isoradial graphs beyond the rational case

The dimer spectral theorem by Kenyon and Okounkov gives a bijection between weights on dimers for periodic planar bipartite graphs modulo elementary transformations and Harnack curves with a standard divisor. Fock constructed an explicit inverse (from a curve to dimer weights) by constructing a (periodic) Kasteleyn matrix using Riemann theta functions of that curve. George proved later a similar spectral theorem for periodic planar electric networks, which involved constructing a (periodic) Laplace operator, whose coefficients (the conductances) are expressed in terms of Prym theta functions. The formulas for conductances and the Kasteleyn matrix make sense more generally for infinite „isoradial" graphs. In that general setup, we prove a relation between the Laplace operator and the Kasteleyn matrix, yielding connections between elementary transformations on graphs for both dimers and electric networks and some identities on theta functions. In the periodic case, this creates a bridge between the two spectral theorems discussed above. We give also an explicit, local expression for the associated Green function, generalizing the work of Kenyon on isoradial graphs with trigonometric weights.
This is a joint work with David Cimasoni (Geneva) and Béatrice de Tilière (Paris Dauphine).

Béatrice de Tilière (University Paris–Dauphine PSL)

Fock's dimer model on the Aztec diamond

We consider the dimer model, or equivalently domino tilings, on the Aztec diamond, and suppose that edges are assigned Fock's weights. The main goal of this talk is to give a compact, explicit formula for the inverse Kasteleyn matrix, thus extending in this very general context previous results of the same kind; in particular, this gives an explicit expression for Boltzmann probabilities. Then, we will prove that the partition function admits a product form, and show how to recover Stanley's celebrated formula as a specific case. Finally, we will show how our expression for the inverse Kasteleyn matrix allows to recover results about limit shapes.
This is based on joint work with Cédric Boutillier.

Adam Doliwa (University of Warmia and Mazury)

Non-commutative bi-rational maps satisfying Zamolodchikov equation, and quadrilateral lattices

I would like to present new solutions of the functional Zamolodchikov tetrahedron equation in terms of birational maps in totally non-commutative variables. The maps are derived from the quadrilateral lattice construction and can be described using the $(20_3, 15_4)$ configuration. We provide also an interpretation of the maps within the local Yang–Baxter equation (matrix refactorization) approach. We exploit decomposition of the maps into the normalization and Veblen maps (spider and resplit moves in dimer terminology) which satisfy the pentagonal condition and are matched by the ten-term relation.

Alexander Fairley (TU Berlin)

Koenigs nets and Laplace sequences

Koenigs nets are parametrised surfaces that are also known as conjugate nets with equal Laplace invariants. We present a symmetry concerning two complementary discretisations of Koenigs nets. The symmetry provides insight on Koenigs nets with terminating Laplace sequences. We focus on a particular class of Koenigs nets with terminating Laplace sequences. We discretise a theorem concerning Koenigs nets with parameter lines that are contained in d-dimensional projective subspaces of $2d$-dimensional projective space.

Vladimir Fock (Université de Strasbourg)

Beyond the Poncelet Porism

The first discrete integrable system — the Poncelet porism — was discovered 202 years ago. We will present several points of view on this system allowing to generalize it to a large class of discrete integrable systems and show its connections to some continuous ones like KdV, SG and KP. The phase space of such systems can be interpreted as the space of pairs of planar curves and line bundles on them, as the space of weights on a periodic dimer lattice, and as a space of configurations of flags in infinite dimensional space subject to certain symmetries. These approaches allows to explicitly solve these systems in terms of theta functions as well as to study their algebraic properties.

Sergey Fomin (University of Michigan)

Incidence geometry and tiled surfaces

We show that various classical theorems of real/complex linear incidence geometry, such as the theorems of Pappus, Desargues, Möbius, and so on, can be interpreted as special cases of a general result that involves a triangulation of a closed oriented surface, or a tiling of such a surface by quadrilateral tiles. This yields a general mechanism for producing new incidence theorems and generalizing the known ones.

Jonah Gaster (University of Wisconsin–Milwaukee)

Why is the Markov Uniqueness Conjecture hard?

The Markov Uniqueness Conjecture asserts that two simple geodesics on the „modular torus" (that is, the quotient of the hyperbolic plane by the commutator subgroup of the modular group) of equal length differ by an isometry of the surface. Of course, each homotopy class of a simple closed curve inherits a well-defined length; slightly less obviously, it also inherits a well-defined twist number, given by the length along the geodesic representative between orthogonal projections of the boundary to the geodesic from its left and right, respectively. The simple closed curves on the torus are naturally parameterized by their slope, a rational number p/q, and one obtains length- and twist-functions on the rationals. Going back at least to work of Fock (and McShane–Rivin), the length-function is known to have some very pleasant properties: when the hyperbolic length of the p/q-curve is divided by q, the result extends to a continuous, decreasing, convex function on $[0, 1]$. The twist-function, by contrast, is substantially worse — its graph is dense in the square. Because the orbit under the isometry group of a simple geodesic is determined by the length-twist pair, this gives some intuitive answer to the titular question.

Terrence George (UCLA)

Dimers and cross-ratio dynamics

I will discuss how cross-ratio dynamics and similar integrable systems in discrete geometry can be realized as dimer cluster integrable systems. Based on joint work with Niklas Affolter and Sanjay Ramassamy.

Wayne Lam (University of Luxembourg)

Discrete hyperbolic Laplacian

The Laplace operator on a Riemannian manifold is a fundamental tool to study the geometry of the manifold. Inspired by electric networks, Laplacians on graphs are defined with edge weights playing the role of conductance. When the edge weights are constant, the graph Laplacian becomes the combinatorial Laplacian and is known to reveal rich combinatorial information of the graph. Given a graph embedded on a surface, it is natural to consider a geometric Laplacian, where edge weights are adapted to the geometry. For the 1-skeleton graph of a geodesic triangulation on a Euclidean surface, there is a „cotangent formula" relating the edge weights to the Euclidean metric. It is known to connect with various problems, e.g., deformations of circle patterns, Delaunay decomposition, discrete harmonic maps and the Y-Delta transform in graphs. In the talk, we introduce the analogue for hyperbolic surfaces.

Feng Luo (Rutgers University)

Discrete uniformization problem for non-compact polyhedral surfaces

The classical uniformization theorem for Riemann surfaces applies to all compact and non-compact connected surfaces. In the realm of discrete uniformization problem for polyhedral surfaces, progresses have been made for compact surfaces. The major remaining issue is the discrete uniformization problem for non-compact surfaces. The challenges in this new setting include formulating the discrete uniformization problem for non-compact surfaces and determining the precise definition of non-compact polyhedral surfaces. In this talk, we will discuss these problems and some of our recent work on these issues.
This is a joint work with Yanwen Luo.

Paul Melotti (Université Paris–Saclay)

Dimers from octahedral integrable equations

Adler, Bobenko and Suris famously classified all consistent, or integrable, equations on octahedra. One of these equations, also known as dKP equation, was shown by Speyer to correspond to certain dimer models on specific graphs, for some natural initial conditions. We give a similar interpretation for the dSKP equation, which gives rise to a ratio of partition functions of oriented dimers, and we obtain all other equations via limits. Those limits also have a combinatorial sense.
Joint works with Niklas Affolter and Béatrice de Tilière.

Christian Müller (TU Vienna)

Framewoks in static equilibrium

The study of flexible, infinitesimal flexible and rigid finite structures is interesting for architecture, engineering, art and also from a geometrical point of view. We will discuss geometric criteria for infinitesimal flexibility and rigidity. The geometry of cable-bar-linkages in static equilibrium and generalizations thereof will be characterized in terms of maps to suitable spaces. We will demonstrate some relations of this theory to discrete differential geometry.
This is joint work with Oleg Karpenkov, Fatemeh Mohammadi and Bernd Schulze.

Hugo Parlier (University of Luxembourg)

Crossing the line: from graphs to curves

The crossing lemma for simple graphs gives a lower bound on the necessary number of crossings of any planar drawing of a graph in terms of its number of edges and vertices. Viewed through the lens of topology, this leads to other questions about arcs and curves on surfaces. Here is one: how many crossings do a collection of m homotopically distinct curves on a surface of genus g induce? The talk will be about joint work with Alfredo Hubard where we explore some of these, using tools from the hyperbolic geometry of surfaces in the process.

Helmut Pottmann (TU Vienna)

From area-preserving Combescure transformations to quad mesh mechanisms

Flexibility is known for many beautiful and surprising results, but also for the difficulties connected with the systematic creation of models, especially for applications such as transformable designs in art and architecture. We focus on transformable designs in form of quad meshes that act as mechanisms under the assumption of rigid faces and hinges in the edges. Q-nets of this type have been studied by Schief, Bobenko and Hoffmann from the perspective of integrability. The basic building blocks are flexible 3×3 sub-meshes, which have been classified by Izmestiev. However, so far these results did not lead to solutions for transformable design. In view of the difficulties of the problem, we first study the counterparts in isotropic geometry, which may be seen as structure-preserving simplification of Euclidean geometry. With the right definition of isotropic isometries between surfaces, one obtains a rich and interesting theory with many counterparts to Euclidean isometries. Metric duality in isotropic space turns flexible Q-nets into Q-nets that admit a continuous family of area-preserving Combescure transformations. These nets can be completely classified, yielding just two different classes. The nets in the first class are special cases of cone nets that have been recently studied by Kilian, Müller and Tervooren. The second class consists of Koenigs nets having a Christoffel dual with the same areas of corresponding faces. This results in constructions of Q-nets of arbitrary mesh size which are isotropic mechanisms. Using these nets to initialize optimization algorithms to turn them into Euclidean mechanisms yields much better results than expected.
This is joint work with Mikhail Skopenkov, Olimjoni Pirahmad, Christian Müller and Caigui Jiang.

Pavlo Pylyavskyy (University of Minnesota)

Coherent pairs of vector-relation configurations.

We study a geometric model associated with bipartite graphs. A state consists of a choice of a vector at each white vertex made in such a way that the vectors neighboring each black vertex satisfy a linear relation. Time evolution is obtained by applying certain local moves. For different choices of the graph one recovers notable dynamical systems including the pentagram map, Q-nets, and discrete Darboux maps. One can also consider a dual setting where a choice of a covector/hyperplane at each black vertex is made in such a way that the covectors neighboring each white vertex satisfy a linear relation. A pair of a vector configuration and a covector configuration associated with the same graph can be compatible, in which case we say they form a coherent pair of configurations. We show that coherence is preserved by the local moves. Translated into geometric language this yields theorems about preservation of compatibility in notable dynamical systems. For example one recovers 3D consistency of Laplace-Darboux dynamics in this way.
This is a joint work with Anton Izosimov, parts of the talk are also based on joint works with Sergey Fomin and with Niklas Affolter, Max Glick, and Sanjay Ramassamy.

Andrew Sagemann-Furnas (North Carolina State University)

Discrete Bonnet Pairs and Minimal Surfaces from Rhombic Embeddings

We develop spin transformations for quad nets in Euclidean 3-space. An invariant spin cross-ratio generalizes the complex cross-ratio and leads to a novel definition for non-planar quad nets in conformal coordinates. As a first application, we define discrete Bonnet pairs as spin transforms of discrete isothermic surfaces. Each pair has the same spin metric and mean curvature. Moreover, we characterize these Bonnet pair nets in terms of special conformal coordinates. As a second application, we present a generalized Weierstrass representation. The simplest examples arise from factorizing cross-ratio systems, leading to discrete minimal surfaces. This recovers the known real factorizing cross-ratio setting and extends it to complex factorizing cross-ratios.
This is joint work with Tim Hoffmann and Max Wardetzky.

Wolfgang Schief (UNSW Sydney)

The Q_V equation: Algorithmic derivation of Backlund transformations for lattice equations via decomposability

The notion of decomposability of a discrete equation defined on a quadrilateral lattice is introduced. The phenomenon of the solutions of a lattice equation satisfying a closed equation on sublattices is not new and has been observed, for instance, in connection with the notion of consistency around the cube (CAC) in integrable systems theory. However, here, decomposability is taken a priori as the basic principle of the analysis rather than a property which one observes a posteriori. The decomposability of the master Q_V equation into a 5-point equation (S_V) is demonstrated and, as a direct consequence, Backlund transformations for both S_V and Q_V are derived. This is then applied to the Q_4, cross-ratio and Q_3 equations as special cases. Thereby, it is demonstrated that, in a specific sense, the notion of decomposability is broader than the notion of CAC. In a similar vein, the decomposition of the integrable master dBKP (Miwa) equation into a 14-point equation is discussed.

Richard Schwartz (Brown University)

Rigidity and Flexibility in Pentagram-Like Maps

Quite a lot is known about the algebraic structure behind the pentagram map and its many generalizations. Many of them are discrete, completely integrable systems. At the same time, the geometry of these maps is much less explored. We don't really know what these maps do to polygons. I will sketch two recent geometric results along these lines. One of these results, a rigidity theorem, says that the orbit of a centrally symmetric octagon under the 3-diagonal map remains convex if and only if the polygon is a Poncelet polygon. The other result, a flexibility theorem, exhibits an infinite family of deep diagonal maps which preserve certain spaces of star-shaped (but not necessarily convex) polygons. These results come with some exciting graphics which I will show during the talk.

Mikhail Skopenkov (KAUST)

Lattice models and discrete complex analysis

We introduce new results on lattice models obtained using the ideas of discrete complex analysis. One result is a percolation formula, generalizing the known ones by Cardy and Schramm. Another result is an elementary model of an electron moving along a line, generalizing Feynman's model and reproducing the usual quantum field theory in the continuum limit. The talk is elementary, and no background in physics is required.
Based on joint works with Mikhail Khristoforov, Stanislav Smirnov, and Alexey Ustinov.

Nina Smeenk (TU Berlin)

Discrete Constant Mean Curvature Surfaces

Smooth constant mean curvature and minimal surfaces are famous examples of isothemic surfaces. In this talk I will present a discretization in terms of discrete S-isothermic surfaces. It mimics various smooth properties, such as the relationship to the Christoffel dual and the constant mean curvature property in terms of the Steiner formula. The associated discrete Gauss maps are closely related to orthogonal ring patterns and orthogonal circle patterns. Staring with a smooth cmc surface, there exists a general method on how to construct a discrete analogue. Using this method will find examples of discrete cmc and minimal surfaces in $\mathbb{R}^3$. Additionally examples of spacelike cmc and maximal surfaces in Lorentz-Minkowski space $\mathbb{R}^{2,1}$ can be obtained.
This is a joint work with Alexander Bobenko and Tim Hoffmann.

Yousuf Soliman (Caltech)

Conservation Laws for Discrete Willmore Surfaces

In this talk, we introduce discrete Willmore surfaces, defined as critical points of the Möbius invariant discretization of the Willmore energy, and derive discrete conservation laws associated to the Möbius symmetry. To describe Möbius equivalence classes of discrete surfaces, we define Möbius structures on simplicial surfaces. These Möbius structures provide a notion of a Cartan geometry for simplicial surfaces and generalize complex cross ratio systems to higher dimensions. For functionals that are already defined on Möbius structures (i.e., the Willmore energy), Euler-Lagrange equations can be computed by working in this larger space, treating integrability as a constraint. From this process we arrive at conserved quantities as well as a geometric description of the Euler-Lagrange equations. These conserved quantities have applications to the design of surfaces in computer graphics. We also describe some deformations of simplicial surfaces using these structures.

Jannik Steinmeier (TU München)

From 4D cross-ratio systems to constant curvature surfaces

Cross-ratio systems are known to be closely related to discrete surfaces of constant curvature. For example, one can find discrete pseudospherical surfaces (K-nets) in the associated family of special cross-ratio systems. Also, using the discrete DPW-method one can construct discrete CMC surfaces from a cross-ratio system, which is interpreted as a discrete holomorphic map. We present reductions of the cross-ratio system which have such discrete surfaces in their associated families. The DPW method then corresponds to taking a slice in a 4D cross-ratio system. Other 4D cross-ratio systems allow for the construction of lattices of breather transformations of pseudospherical surfaces.

Gudrun Szewieczek (TU München)

Discrete isothermic nets with a family of spherical parameter lines from holomorphic maps

Smooth surfaces foliated by a family of planar or spherical curvature lines are an active area of research, driven by both purely differential geometric aspects and practical applications such as architectural design. In integrable geometry it is a natural question to ask which of these surfaces admit a conformal curvature line parametrization and are therefore isothermic surfaces. It is an open problem to explicitly describe all those smooth isothermic surfaces. However, over time, prominent examples were found in this rich integrable surface class: above all Wente's torus. More recently, further specific examples have led to the discovery of compact Bonnet pairs and to free boundary solutions for minimal and CMC-surfaces.This talk covers a discrete version of the problem: we shall generate all discrete isothermic nets with a family of spherical curvature lines from special discrete holomorphic maps via the concept of „lifted-folding". In particular, we point out how this novel approach leads to quasi-periodic solutions and to topological tori with symmetries.
This is joint work with Tim Hoffmann.

Sergei Tabachnikov (Penn State)

A family of maps and a vector field on the space of plane polygons

I will report on theoretical and experimental study of a 1-parameter family of transformations and their limiting vector field on the space of plane polygons. These transformations are discrete analogs of completely integrable transformations of closed plane curves, known as the bicycle correspondence, that are geometric realizations of the Backlund transformation of the planar filament equation. In the case of odd-gons, these transformations are symplectic on the quotient space of polygons by parallel translations, and the vector field is Hamiltonian. In the case of triangles, the vector field is completely integrable and, conjecturally, so are the transformation.
This is joint work with M. Arnold and L. Costa.

Jan Techter (TU Berlin)

Koenigs binets

A (smooth) Koenigs net is a conjugate net with equal Laplace invariants. Koenigs nets are a projecively invariant generalization of isothermic nets. Binets are maps from the vertices and faces of $\mathbb{Z}^2$. We introduce a discretization of Koenigs nets as binets with equal Laplace invariants. Koenigs binets generalize the classical notion of „vertex based" Koenigs nets by Bobenko and Suris, and „face based" Koenigs nets by Doliwa. We show that Koenigs binets, analogous to smooth Koenigs nets, admit Christoffel dual binets. Finally, we consider N-dimensional Koenigs binets on the vertices and faces of $\mathbb{Z}^N$, $N > 2$, and show that Koenigs binets are multi-dimensionally consistent.

Alexander Veselov (Loughborough University)

Delay Painleve-I equation and Masur-Veech volumes

The delay Painleve-I equation is obtained as the simplest delay periodic reduction of Shabat's dressing chain. We use it to generate a new family of Bernoulli–Catalan polynomials and apply them to the calculation of the Masur–Veech volumes of the moduli spaces of meromorphic quadratic differentials.
The talk is based on a joint work with John Gibbons and Alex Stokes.