Discretization in geometry and dynamics 2022

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Discretization in Geometry and Dynamics

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The dSKP recurrence: combinatorial aspects and geometric systems

Béatrice de Tilière

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The subject of this talk is the discrete Schwarzian Kadomtsev-Petviashvili (dSKP) recurrence.

First, we will prove an explicit expression for the solution as a function of the initial data; more precisely we will show that the solution is the ratio of two partition functions of an associated oriented dimer model. There are some cancellations in each partition function, and we will show an alternative, cancellation free expression involving complementary trees and forests. Apart from its combinatorial interest, this is used to prove singularity results.

Next, we will show how this equation appears in different geometric systems as for example: discrete holomorphic functions, polygon recutting or the pentagram map. Using our previous results, we prove explicit expressions for their solution, and handle the reoccurrence of singularities, known as the Devron property.

This is based on joint work with Niklas Affolter and Paul Melotti.

Random dimer coverings with doubly periodic edge weights

Tomas Berggren

Massachusetts Institute of Technology

This talk will be centered around domino tilings, or dimer coverings, of the Aztec diamond with doubly periodic edge weights. Such model exhibits a rich structure, for instance, in the limit three types of regions may appear, the frozen, rough and smooth regions (also known as solid, liquid and gas regions). We will discuss asymptotic results, both on the macroscopic and microscopic scale.

The asymptotic results rely on an expression of the correlation kernel in terms of a Wiener-Hopf factorization of a matrix valued function, which is defined in terms of the edge weights. We will discuss how such Wiener-Hopf factorization can be obtained in a form that are suitable for asymptotic analysis.

Circles, spherical curvature lines, and (M-)Lie inversions

Gudrun Szewieczek

University of Vienna

Smooth surfaces and orthogonal coordinate systems with circular or spherical coordinate lines have been intensively studied in classical surface theory. Due to their well-structured but not trivial geometry those give interesting examples and subclasses that have led to remarkable results; for example Wente torus, compact Bonnet pairs and Dupin cyclidic metrics.

In this talk we shall discuss discrete analogs of surfaces with a family of spherical/planar curvature lines and cyclic orthogonal coordinate systems. In particular, we will demonstrate that Lie sphere geometry provides an efficient and simple way to generate them by appropriately evolving some initial data. Key to success are (M-)Lie inversions, the basic transformations of Lie sphere geometry that generalize the concept of reflections in planes and spheres.

Circular nets with spherical parameter lines and terminating Laplace sequences

Alexander Fairley

Project C01, Technische Universität Berlin

In the context of discrete differential geometry, surfaces with spherical curvature lines are classical surfaces that motivate the study of circular nets with spherical parameter lines. An important feature of surfaces with spherical curvature lines is that they have terminating Laplace sequences. There is a similar phenomenon for circular nets with spherical parameter lines

Geometry processing for digital twinning

Pierre Alliez

French National Institute for Research in Digital Science and Technology

Geometry Processing is a field of research aiming to design and implement efficient data structures and algorithms for the acquisition, reconstruction, analysis and transmission of 3D models and scenes.

In the first part of my talk, I will explain how the current enthusiasm for digital twins is posing a range of scientific challenges such as continuous acquisition and 3D reconstruction, physics-informed geometric modeling and cognitive 3D models.

In the second part of my talk, I will review some recent contributions utilized for digital twins of factories, civil infrastructures and industrial facilities. I will then detail a novel algorithm entitled "Alpha wrapping with an offset" that is guaranteed to convert a raw input 3D geometry into a watertight, orientable surface triangle mesh that strictly encloses the input. Such a conservative standpoint is primarily motivated by robotic simulation applications such as collision avoidance or trajectory optimization. The algorithm is made generic via an abstract interface probing the input, making it possible to wrap a heterogeneous set of 3D primitives.

Self-assembly of proteins: a 2D model case

Lukas Mayrhofer

Project A13, Technische Universität München

The coat proteins of the tobacco mosaic virus self-assemble into a helical shell. Such large-scale processes cannot currently be replicated using all-atom molecular dynamics simulations. To simulate self-assembly, we instead explore a more coarse-grained approach.

Our approach consists of two main components: A model for the solvation free energy and a suitable stochastic descent method such as Metropolis-Hastings or hybrid Monte Carlo that drives the simulation by minimizing this energy. The solvation free energy is modelled, inspired by Hadwiger's integral theorem, as a linear combination of quantities that only depend on the shape of the protein involved in the self-assembly. This model allows fast evaluation of the energy.

As a proof of concept, we simulate the dynamics of identical copies of an asymmetric 2D model shape, which are observed to robustly self-assemble into a unique final state.

Approximate and discrete vector bundles in theory and applications

Luis Scoccola

Northeastern University

In the last decades, several successful approaches to the problem of estimating the homology of an embedded manifold from finite samples have been proposed. Motivated by the problem of estimating further invariants of manifolds, such as characteristic classes, I will introduce a theory of approximate and discrete vector bundles which includes algorithms for the effective computation of low-dimensional characteristic classes from finite samples. I will also describe an application to the problem of reducing the dimensionality of topologically complex data while preserving its large scale topology. No previous knowledge of the theory of vector bundles will be assumed.

The genetic column generation algorithm for multi-marginal optimal transport

Maximillian Penka

Project B08, Technische Universität München

We extended the recently introduced genetic column generation algorithm for high-dimensional multi-marginal optimal transport from symmetric to general problems. In my presentation, I first will introduce the multi-marginal optimal transport problem, starting from the well-known two-marginal case. The theoretical focus will be on sparsity results, which are essential for the following part. In the second part I will explain the Genetic Column Generation Algorithm and show its applications in the context of Wasserstein barycenters.

Polynomial recurrences from discrete (rational) dynamical systems

Claude Michel-Viallet

Theoretical and High Energy Physics Laboratory (LPTHE), Sorbonne University and CNRS

The analysis of the iterates of discrete time dynamical systems gives, in the case of rational (actually birational) evolution, a good test of their integrability, but it also produces remarkable sequences of polynomials. These polynomials are generated by "Somos-like" sequences, enjoying a generalized Laurent property. We will explain and exemplify this process.

A 3D generalization of QRT maps and integrable Kahan discretization

Kangning Wei

Project B02, Technische Universität Berlin

Introduced in 1988, the QRT map is one of the most popular discrete planar integrable systems, which has a rather simple construction and rich geometric properties. We develop a framework that generalizes QRT maps and QRT roots to 3D, which allows us to create new integrable maps as a composition of two involutions. We show that under certain geometric conditions, the new maps become of degree 3.

The motivation of our construction comes from the problem of integrable discretizations. We use this construction to create new families of discrete integrable maps and in particular, we solve the problem of finding integrable discretizations of the Zhukovski- Volterra gyrostat with two βs.

This is joint work with Yuri Suris and Jaume Alonso.

Inference in dynamical systems and model reduction

Sayan Mukherjee

Max Planck Institute for Mathematics in the Sciences, Leipzig

Under what conditions are Bayesian methods for inference in deterministic or stochastic dynamical systems with observational noise consistent?

We will highlight the relation between Bayesian inference and a classical idea in dynamical systems called the thermodynamic formalism. We will provide a partial answer a variational characterization of a partition function that answers the question of what conditions allow for inference. We also provide asymptotic results concerning (generalized) Bayesian inference for certain dynamical systems based on a large deviation approach.

The proposed framework is quite general, we apply it to two very different classes of dynamical systems: continuous time hyper-mixing processes and Gibbs processes on shifts of finite type. We also show that the generalized posterior distribution concentrates asymptotically on those parameters that minimize the expected loss and a divergence term, hence proving posterior consistency. We close with a discussion of model reduction in dynamical systems.

Filament based plasma: modeling solar phenomena using curves

Marcel Padilla

Project C07, Technische Universität Berlin

Simulation of stellar atmospheres, such as that of our own sun, is a common task in CGI for scientific visualization, movies and games. A fibrous volumetric texture is a visually dominant feature of the solar corona - the plasma that extends from the solar surface into space.

These coronal fibers can be modeled as magnetic filaments whose shape is governed by the magnetohydrostatic equation. The magnetic filaments provide a Lagrangian curve representation and their initial configuration can be prescribed by an artist or generated from magnetic flux given as a scalar texture on the sun's surface. Subsequently, the shape of the filaments is determined based on a variational formulation. The output is a visual rendering of the whole sun. We demonstrate the fidelity of our method by comparing the resulting renderings with actual images of our sun's corona.

Reference: M. Padilla, O. Gross, Felix Knöppel, A. Chern, U. Pinkall, P. Schröder. *Filament based plasma.* ACM Transactions on Graphics. 2022, 41(4), 1–14. DOI:10.1145/3528223.3530102

Existence of periodic bounce orbits

Peter Albers

University of Heidelberg

Periodic bounce orbits are generalizations of billiard orbits. I will present an existence result for periodic bounce orbits for any smooth (not necessarily convex) table in any dimension using variational methods going back to Benci-Giannoni combined with Rabinowitz Floer homology. This is joint work with Gabriele Benedetti, Marco Mazzucchelli, and Anna-Maria Vocke.

A discrete-to-continuum Γ-limit for a discrete two-dimensional frustrated spin system

Melanie Koser

Project B11, Humboldt Universität Berlin

We consider magnetic compounds whose atoms are ordered in a regular crystalline structure and associate to each atom its so-called *spin*, a unit vector in $\mathbb{R}^2$. Complex geometric structures in the spin field may be the result of the competition between *anti*- or *ferromagnetic* interactions. In ferromagnetic materials, spins prefer to be aligned, whereas in anti-ferromagnetic compounds one cannot observe a global orientation of the spins. The competition between these two interactions leads to *frustration mechanisms* in the system. We consider the lattice energy of certain materials, in which anti-ferromagnetic (AF) and ferro- magnetic (F) interactions coexist, and are modeled by the $J_1 - J_3$ F-AF model on a square lattice.

Our aim is to analyze minimizers and their properties. Apart from scaling laws for the minimal energy, a helpful tool can be a Γ-convergence result for a suitable sequence of the discrete energies towards a continuous energy. Having local compactness additionally, one knows that a sequence of minimizers converges towards a minimizer of the Γ-limit. Therefore, one can analyze the properties of the minimizers of the Γ -limit to obtain properties of the discrete minimizers. In this talk, we present a discrete and a continuous scaling law and an insight on the key ingredients for the proof of the Γ -convergence.

Towards an algorithm for identifying the global dynamics of multi-parameter systems of ordinary differential equations

Konstantin Mischaikow

Rutgers University

We will discuss potential first steps towards the long term of providing an algorithm for identifying the global dynamics of multi-parameter systems of ordinary differential equations (ODE). Even though we are far from achieving this goal this initial effort require a wide variety of techniques and arguments, and thus we will limit ourselves to few major points:

(1) What do we mean by solve? We will argue that a non-traditional notion of solution is necessary and suggest one based on order theory and algebraic topology.

(2) How should the nonlinearity and parameter space of the ODE be discretized? We will introduce a specific family of differential equations (ramp systems) for which we can produce a combinatorial representation of the dynamics and a well defined finite decomposition of parameter space.

(3) We will provide a description of the analytic challenges of proving that the homological characterization of the dynamics is correct.

(4) We will provide a few simple examples of the computations that can be done currently.

This is ongoing work with W. Duncan, D. Gameiro, M. Gameiro, T. Gedeon, H. Kokubu, H. Oka, B. Rivas, and E. Vieira.

Homology inference for the degree-Rips bifiltration

Alexander Rolle

Project C04, Technische Universität München

The degree-Rips bifiltration is a density-sensitive construction based on the Vietoris-Rips filtration. I will present an example, motivated by experiments in a recent paper of Blumberg and Lesnick. They consider a point cloud concentrated on an annulus in the plane, and another obtained by adding outliers. Then they compare the homology of the degree-Rips complexes of both point clouds, and they observe that the relationship between these is not well explained by existing results on the stability of degree-Rips. Using the Adamaszek-Adams computation of the Vietoris-Rips complexes of the circle, I compute the homotopy types of the degree-Rips complexes of several metric probability spaces connected to these experiments. Motivated by these results, I suggest a homology inference approach to degree-Rips and related constructions.

Discrete surfaces via binets: Geometry

Jan Techter

Project A02, Technische Universität Berlin

In several classical examples discrete surfaces naturally arise as pairs consisting of combinatorially dual nets describing the “same” discrete surface.

On the basis of this observation we introduce a discretization of parametrized surfaces via binets, which are maps from the vertices and faces of $\mathbb{Z}^2$ into $\mathbb{R}^3$. We take a closer look at discrete principal binets, which generalize the notions of circular and conical nets, and appear in examples such as orthogonal circle patterns, orthogonal ring patterns, and discrete confocal quadrics.
Discrete principal binets admit a natural discrete Gauss map, a lift to Möbius geometry, Laguerre geometry, and Lie geometry.

Moreover, we introduce discrete Koenigs binets in terms of equal Laplace invariants. They admit Christoffel duals and generalize the classical notion of "vertex based" Koenigs nets by Bobenko and Suris, and “face based” Koenigs nets by Doliwa. Together the notions of discrete principal binets and discrete Koenigs binets give rise to discrete isothermic binets, which turn out to coincide with discrete isothermic nets based on checkerboard patterns as introduced by Dellinger.

Discrete surfaces via binets: consistency

Niklas Affolter

Project A01, Technische Universität Berlin

We consider generalizations of binets defined on the vertices and faces of $\mathbb{Z}^N$ for $N > 2$. We need two new types of multi-dimensionally consistent nets: plane nets and line compounds. First, we consider pairs of a conjugate net and a plane net. As a special case, we obtain a definition of principal binets on $\mathbb{Z}^N$. It turns out that the Möbius lift of a principal binet is a polar binet, which leads to a simple proof of the consistency of principal binets. Second, we consider pairs of a conjugate net and a line compound. As a special case, we obtain a definition of Koenigs binets on $\mathbb{Z}^N$. As on $\mathbb{Z}^2$ a particular case of Koenigs binets on $\mathbb{Z}^N$ are pairs of Bobenko-Suris Koenigs nets and Doliwa Koenigs nets. As a by-product, we show that line compounds provide a setup to obtain a consistent generalization of Doliwa Koenigs nets to $\mathbb{Z}^N$.

Digital halftoning via weighted sigma-delta quantization

Anna Veselovska

Project C02, Technische Universität München

In this presentation, we consider error diffusion techniques for digital halftoning from the perspective of 1-bit ΣΔ quantization [2]. We introduce a method to generate ΣΔ schemes for two-dimensional signals as a weighted combination of its one-dimensional counterparts and show that various error diffusion schemes proposed in the literature, as e.g. [1], can be represented in this framework via ΣΔ schemes of first order.

Under the model of two-dimensional bandlimited signals, which is motivated by a mathematical model of human visual perception, we derive quantitative error bounds for such weighted ΣΔ schemes. Motivated by the correspondence between existing error diffusion algorithms and first-order ΣΔ schemes, we study the performance of the analogous weighted combinations of second-order ΣΔ schemes and show that they exhibit a superior performance in terms of guaranteed error decay for two-dimensional bandlimited signals. In numerical simulations for real world images, we demonstrate that with some modifications to enhance stability this high-quality performance also translates to the problem of digital halftoning.

This is joint work with Prof. Dr. Felix Krahmer from the Technische Universität München.

References:

[1] R.W. Floyd, L. Steinberg. *An adaptive algorithm for spatial grey scale.*. Proceedings of the Society of Information Display. 1976, 17, 75–77

[2] C. S. Güntürk. *One-bit sigma-delta quantization with exponential accuracy.* Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences. 2003, 56(11), 1608-1630.

[3] F. Krahmer, A. Veselovska. *Enhanced Digital Halftoning via Weighted Sigma-Delta Modulation.* arXiv preprint. 2022, arXiv:2202.04986