One-bit sigma-delta modulation on a closed loop

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SFB TRR 109 | Discretization in Geometry and Dynamics

Overview of the problem

Our task:

- For K-bandlimited signals whose domain is $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, perform 1-bit $\Sigma\Delta$ modulation

Why is it challenging?

- $\Sigma\Delta$ modulator requires causality, hence we have mismatch at the initialization point and consequently large reconstruction error — need to fix that!

Why is it useful?

- If domain is extended from 1D to 2D closed manifolds, 1-bit $\Sigma\Delta$ modulation can be used as a digital halftoning technique for 3D color printing

1-bit ΣΔ modulation

First order 1-bit $\Sigma\Delta$ scheme:

$$
\begin{aligned}
v_n &= v_{n-1} + y_n - q_n & y_n &\text{ - samples (input)} \\
q_n &= \operatorname{sign}(v_{n-1} + y_n) & q_n &\text{ - quantized values (output)} \qquad (1) \\
\text{initialize } v_{-1} &:= 0 & v_n &\text{ - state variables}
\end{aligned}
$$

Assumptions for signal f:

- $f \in L^2(\mathbb{T})$, i.e., f is 2π-periodic
- $\|f\|_\infty \le 1$
- f is K-bandlimited, i.e., $\widehat{f} \in [-K, K]$
- samples $y_n = f\left(\frac{2\pi n}{N}\right)$, $n = 0, \dots, N-1$, where $N = 2\lambda K + 1$ with oversampling parameter $\lambda \geqslant 1$

Signal reconstruction

Sampling theorem & reconstructed signal:

$$
f(t) = \frac{1}{N}\sum_{n=0}^{N-1} y_n \varphi_K\left(t - \frac{2\pi n}{N}\right), \qquad f_{rec}(t) = \frac{1}{N}\sum_{n=0}^{N-1} q_n \varphi_K\left(t - \frac{2\pi n}{N}\right),
$$

where the reconstruction kernel is the Dirichlet kernel

$$
\varphi_K(\xi) = \frac{\sin((2K+1)\frac{\xi}{2})}{\sin(\frac{\xi}{2})}
$$

Reconstruction error:

$$
\begin{aligned}
|f(t) - f_{rec}(t)| &= \frac{1}{N}\left|\sum_{n=0}^{N-1}(y_n - q_n)\varphi_K\left(t - \frac{2\pi n}{N}\right)\right| \\
&\le \frac{1}{N}\|\varphi_K'\|_{L^1} + \underbrace{\frac{|v_{N-1}|}{N}}_{\le \frac{1}{N}} \underbrace{\|\varphi_K\|_\infty}_{=2K+1}
\end{aligned}
$$

Can we get rid of the second term in the above estimate? Yes!

Proposed modification of ΣΔ scheme

- **Aim:** get rid of additional error term $\frac{|v_{N-1}|}{N}\|\varphi_K\|_\infty$
- **Idea:** run $\Sigma\Delta$ for updated samples $\tilde{y}_n := y_n + \delta$, where $\delta = -\frac{v_{N-1}}{N}$ (2)
  and reconstruct from resulting output $\tilde{q}_n$ using the Dirichlet kernel

Theorem (Krause-Solberg, Graf, Krahmer)

Using 1-bit $\Sigma\Delta$ modulation scheme (1) on updated samples $(\tilde{y}_n)_n$ leads to $(\tilde{v}_n)_n$ with $\tilde{v}_{N-1} = 0$.

Reconstruction error after modification

From updated samples we get

$$
\begin{aligned}
|f(t) - \tilde{f}_{rec}(t)| &= \frac{1}{N}\left|\sum_{n=0}^{N-1}(y_n - \tilde{q}_n)\varphi_K\left(t - \frac{2\pi n}{N}\right)\right| \\
&\le \frac{2}{N}\|\varphi_K'\|_{L^1} + \underbrace{\frac{|\tilde{v}_{N-1}|}{N}}_{=0}\|\varphi_K\|_\infty + \underbrace{\delta}_{\le\frac{1}{N}}
\end{aligned}
$$

Numerical experiments & discussion

Consider signal $f(t) = 0.1\sin(5t)\cos(10t) + 0.2$ with bandwidth $2K = 30$.

Reconstruction error for first order $\Sigma\Delta$ scheme

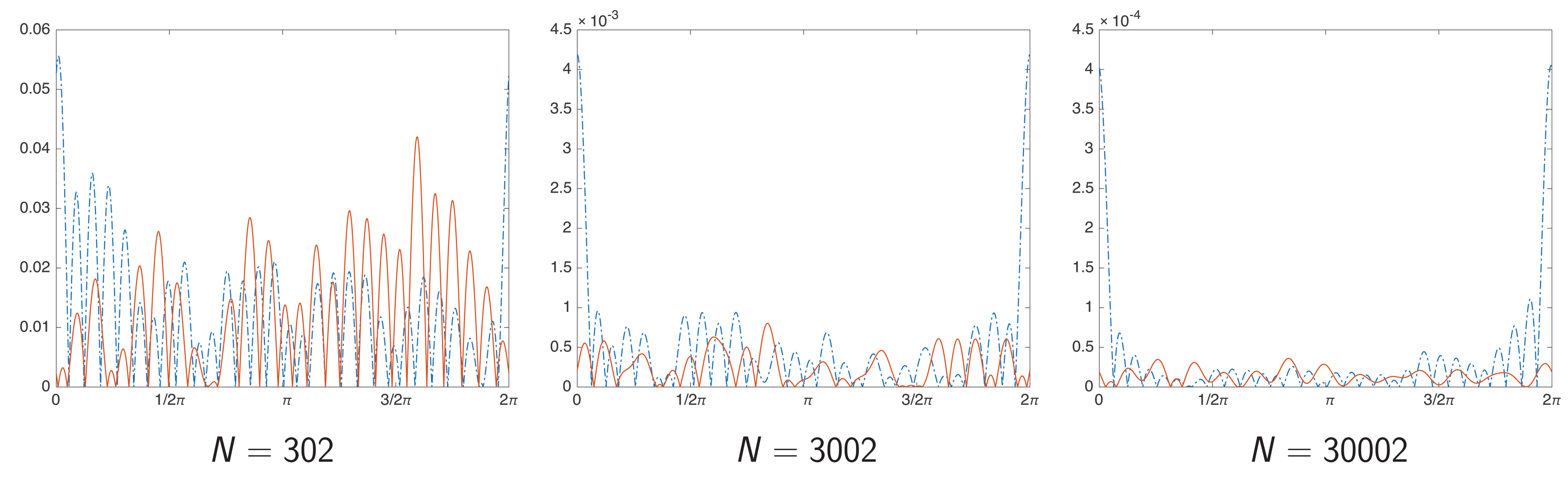

Figure 1. Reconstruction error for classical first order scheme (dashed) and for modified scheme (solid)

- Error is now distrubuted evenly, no large error at initialization point
- L^∞ norm of the error is reduced

Outlook: Reconstruction error for higher order $\Sigma\Delta$ schemes [2]

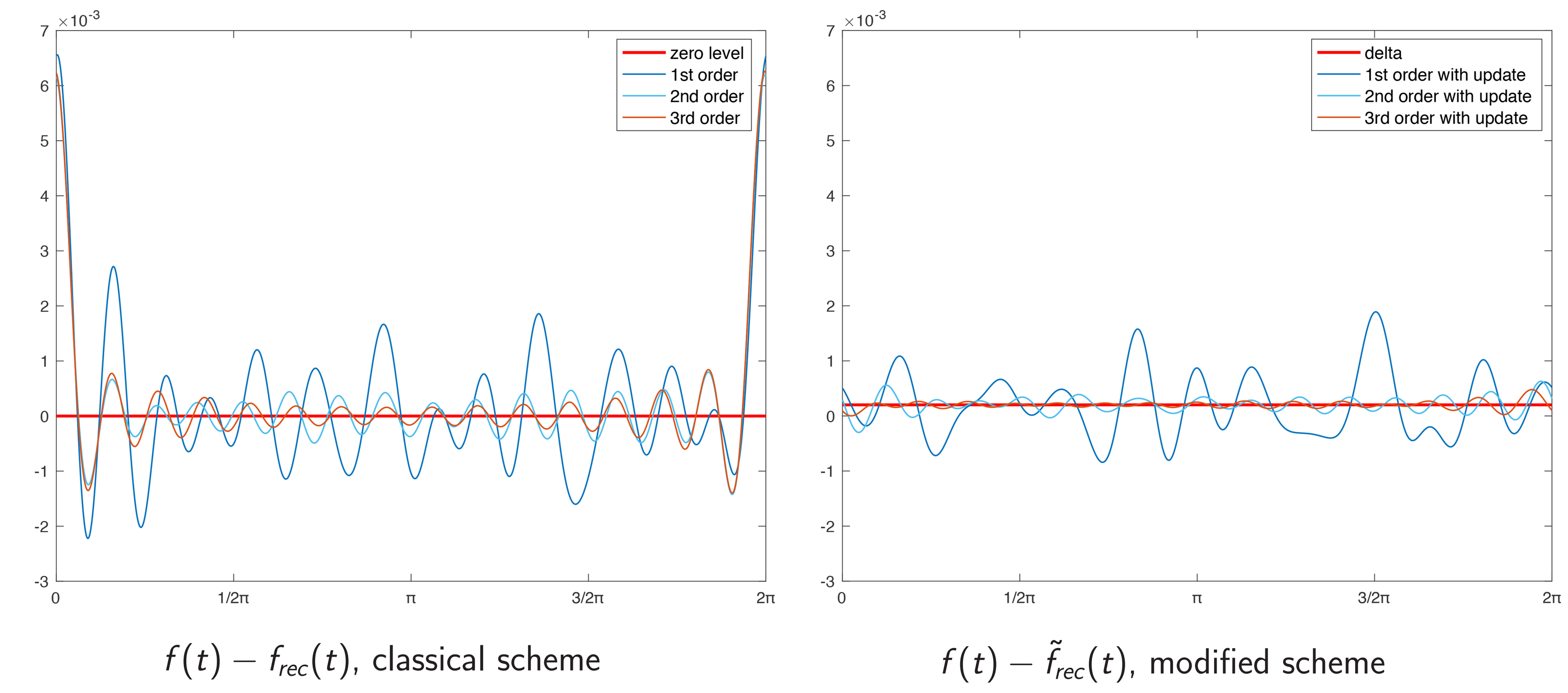

Figure 2. Reconstruction error behavior for classical and for modified schemes

- Updates are higher order variants of (2)
- Error keeps oscillating around δ, constant shifts of $\mathcal{O}(\frac{1}{N})$ cannot be avoided
- Modified scheme optimizes oscillation amplitude around a constant shift, thus leading to less visible artifacts

Long-term goal: Digital halftoning in 3D printing

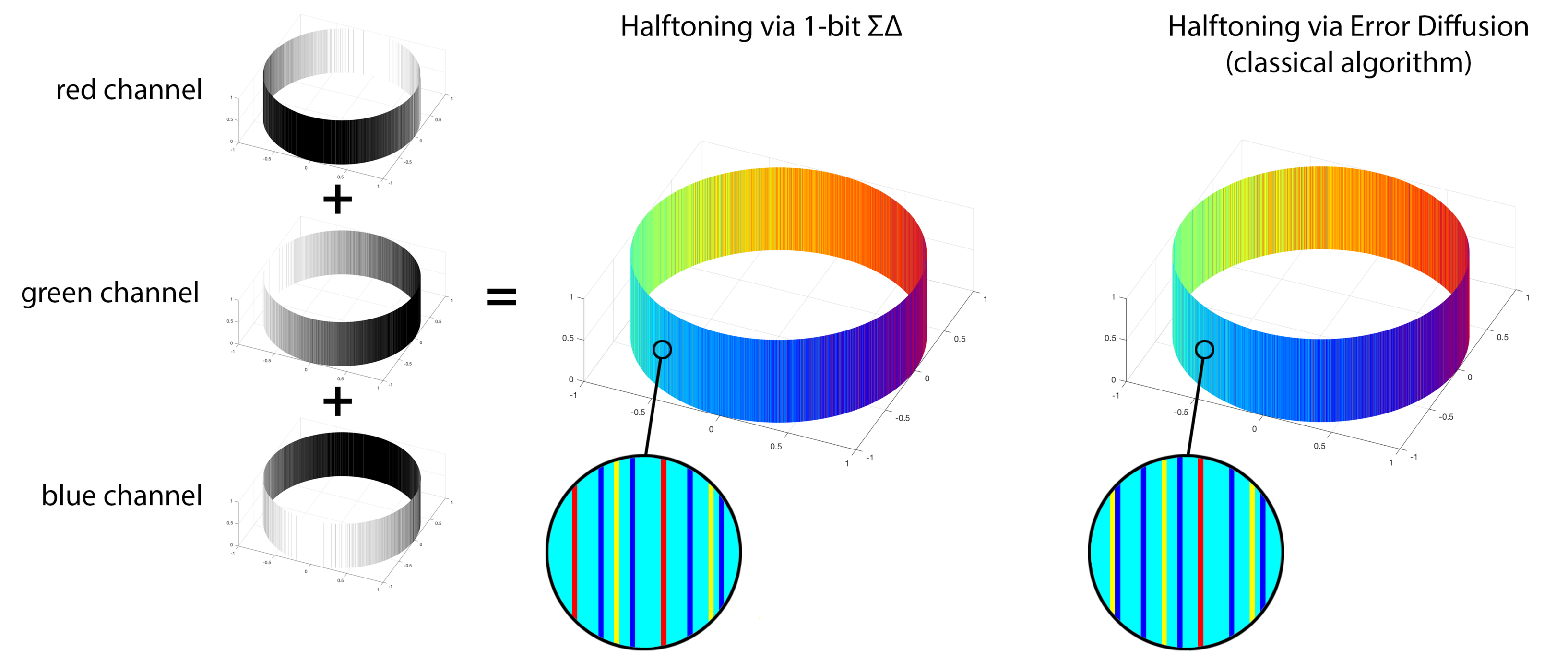

Figure 3. Representation of colors on a surface using digital halftoning

- 1-bit $\Sigma\Delta$ modulation as digital halftoning technique: works well because error is a high-pass sequence while human eye acts as a low-pass filter [3]
- Major challenge in 3D digital halftoning: surfaces to be colored are closed
- 3D digital halftoning can be utilized in 3D printing tecnhologies which rely on voxel-wise printing (e.g. HP Multi Jet Fusion) [4]

References

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