Liftings and stresses for planar periodic frameworks *

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ABSTRACT

We formulate and prove a periodic analog of Maxwell's theorem relating stressed planar frameworks and their liftings to polyhedral surfaces with spherical topology. We use our lifting theorem to prove rigidity-theoretic properties for planar periodic pseudo-triangulations, generalizing their finite counterparts. These properties are then applied to questions originating in mathematical crystallography and materials science, concerning planar periodic auxetic structures and ultrarigid periodic frameworks.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Non-numerical algorithms and problems—*Geometric Problems and Computations*; G.2.2 [**Discrete Mathematics**]: Graph Theory

Keywords

Maxwell's theorem, periodic framework, periodic stress, liftings, periodic pseudo-triangulation, expansive motion, auxetics, ultrarigidity

1. INTRODUCTION

A remarkable correspondence (see Fig. 1) between planar stressed graphs, their duals and polyhedral surfaces with a spherical topology has been established in 1870 by James Clerk Maxwell:

Maxwell's Theorem [21] *A planar geometric graph (G, p) supports a non-trivial stress on its edges iff it has a dual reciprocal diagram iff it has a non-trivial lifting to 3D as a polyhedral terrain.*

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Definitions are given in Section 2. A closely related instance of this theorem is the classical correspondence between Voronoi diagrams, Delauney tesselations, and their 3D lifting onto a paraboloid.

Maxwell's theorem has many applications, more recently in robustness of geometric algorithms, rigidity theory, polyhedral combinatorics and computational geometry [10, 17, 24, 25, 9, 29]. Relevant to our paper is its role in establishing the existence of planar expansive motions used in the solution to the Carpenter's Rule problem [9], and in proving the expansive properties of pointed pseudo-triangulation mechanisms that are central to the algorithm for convexifying simple planar polygons of [28, 29].

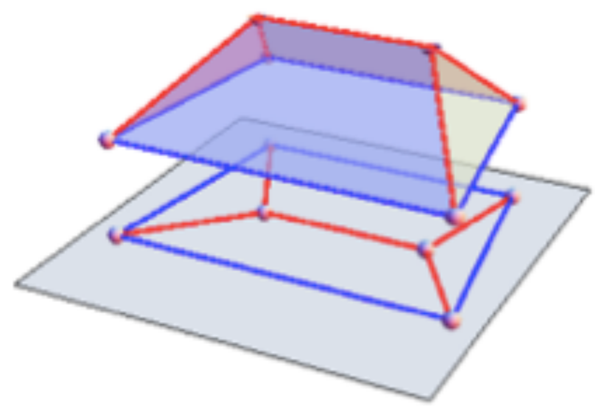
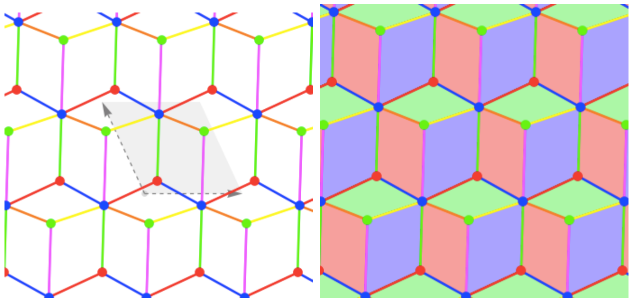


Figure 1: A finite planar stressed graph and a Maxwell lifting.

Our results. In this paper we prove the following *periodic* analog of Maxwell's theorem.

Main Theorem *Let (G, Γ, p, π) be a planar non-crossing periodic framework. A stress induced by a periodic lifting is a periodic stress and conversely, any periodic stress is induced by a periodic lifting, determined up to an arbitrary additive constant.*

The precise definitions, previewed in Fig. 2, will be given in Section 3. Non-crossing periodic graphs can be seen as graphs embedded on the flat torus. However, as it will become clear from our proof, to reason on a fixed torus would be too restrictive a perspective. The most important ingredient that makes possible this result is our recent definition



(a) (b)

Figure 2: (a) A stressed periodic framework. Vertex and edge orbits are similarly colored. (b) Coloring the faces helps visualize its 3D lifting as a periodic arrangement of cubes.

[2] of *periodic rigidity*, which allows the periodicity lattice to deform. The corresponding dual concept of *periodic stress* from [2] turns out to be precisely the notion of stress that is needed for the Main Theorem. It is more constrained than the classical *self-stress* based solely on equilibrium at all vertices; to maintain a proper distinction, we refer to the latter one as an *equilibrium stress*.

This result was motivated by questions from mathematical crystallography and computational materials science. We demonstrate its power with two applications: ultrarigid and auxetic frameworks.

Ultrarigidity of periodic frameworks. Our proof of the correspondence between periodic liftings and periodic stresses will proceed by showing how to obtain a transparent, algebraic matching of all the concepts involved after a sufficient *relaxation of periodicity*. Such relaxations are also essential tools for estimating the *asymptotic behavior* of a periodic framework. Fig. 3 illustrates the concept.

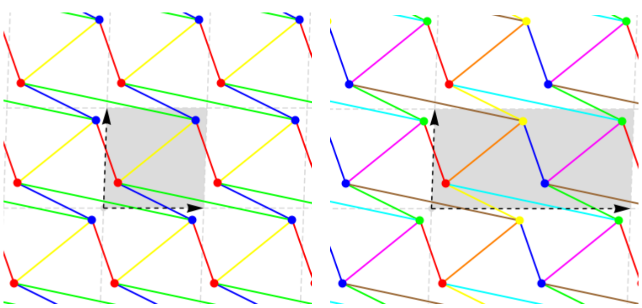


Figure 3: A periodic framework and a 2×1 relaxation of its lattice.

By definition [4], a periodic framework is *ultrarigid* if it is and remains infinitesimally rigid under arbitrary relaxations of periodicity to subgroups of finite index. Our new proof techniques will lead to an infinite family of ultrarigid examples, obtained from:

Periodic pseudo-triangulations. We use the Main Theorem to study a new class of planar non-crossing periodic frameworks called *periodic pointed pseudo-triangulations* or shortly *periodic pseudo-triangulations*. They are a natural analog of the finite version defined and studied in [28, 29] and possess *mutatis mutandis* many outstanding characteristics related to rigidity and deformations [28, 29, 26, 27].

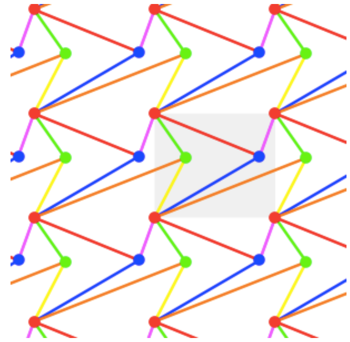


Figure 4: A periodic pseudo-triangulation.

Here we focus on the *expansive* one-degree-of-freedom mechanisms they provide and on the noteworthy property, in the periodic setting, of becoming *ultrarigid* after one edge-orbit insertion.

Deformations: auxetic and expansive behavior. The significance of expansive motions is well recognized in the finite setting [9, 29, 27]: when the distance between any pair of vertices cannot decrease, self-collision of the framework is avoided. Periodic expansive motions have not yet been explored, although a related, yet weaker notion of *auxetic behavior*, has recently attracted a lot of attention in materials science [11, 18, 23]. Since these concepts arise in such different fields, we include a brief and necessarily selective overview of the relations existing between the purely geometric theory pursued in this paper and the larger context explored in crystallography, solid state physics and materials science [19, 30].

The notion of *auxetic behavior* is formulated using the concept of negative Poisson's ratio [12, 11], which relies on physical properties of the material: when two forces pull in opposite directions along an axis, most materials are expected to expand along this axis and to contract along directions perpendicular to it. Auxetic behavior refers to the rather

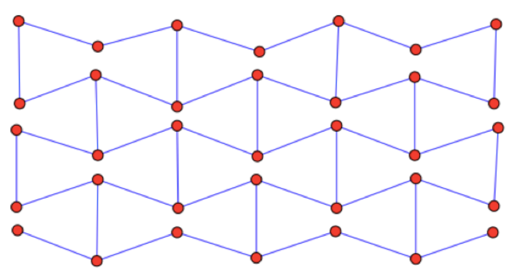


Figure 5: The "reentrant" honeycomb is the emblematic auxetic example.

counter-intuitive lateral widening upon application of a longitudinal tensile strain. A purely geometric expression of this behavior is not anticipated in all situations. However, for periodic frameworks, we have recently proposed the general geometric notion of *auxetic path* in the deformation space of the periodic framework [6]. Relying on this formulation, we prove that an expansive deformation path is necessarily an auxetic path. Periodic pseudo-triangulations thus exhibit auxetic behavior and offer an infinite supply of planar examples of "auxetic frameworks". By contrast, only a few, sporadic auxetic periodic examples have appeared in the literature (see Fig. 5).

Organization. In Sect. 2 we formulate the correspondence between liftings and stresses in (finite or infinite) graphs, and introduce the standard example that differentiates between the equilibrium and periodic versions of stress. Sect. 3 specializes these concepts to periodic liftings and stresses. Sect. 4 completes the proof of our Main Theorem by providing the link with periodic rigidity. The theorem is then applied in Sect. 5 to prove that periodic pseudo-triangulations have expansive paths. The connections with auxetic behavior and ultrarigidity conclude the paper.

2. LIFTINGS AND STRESSES

To formulate and prove our Main Theorem, we start with those concepts and properties that do not depend (yet) on periodicity, which is introduced in the next section.

Planar graphs and frameworks. A graph $G = (V, E)$ is given by a (discrete) set of vertices V and a set of edges E. We consider only locally finite, simple, unoriented graphs. A *placement* or (straight-line) realization of G in $\mathbb{R}^d$ is given by a mapping $p : V \to \mathbb{R}^d$ of the vertices to points in $\mathbb{R}^d$, such that the two endpoints of any edge $e \in E$ are mapped to distinct points in $\mathbb{R}^d$ and an edge $\{u, v\}$ is mapped to a segment $[p(u), p(v)]$. We assume that all placements are locally finite maps in $\mathbb{R}^2$ or $\mathbb{R}^3$, and we use the term *planar placement* for $\mathbb{R}^2$, when the distinction is necessary. A *framework* or *geometric graph* (G, p) is a graph G together with a placement p.

A planar placement is *non-crossing* if any pair of edges induces disjoint closed segments, with the possible exception of the common endpoint, if the edges are adjacent. A graph G is *planar* if it admits a non-crossing placement.[1] We will consider only connected graphs, therefore a non-crossing placement (G, p) induces a connected subset of the plane. A *face* U is a connected component of the complement, and is described combinatorially by the cyclic collection of its boundary vertices or edges. When referring to a *planar graph* G, we assume that the choice of face cycles F has already been decided: G now denotes the entire collection $G = (V, E, F)$ of vertices V, edges E *and* face cycles F. We assume that the boundary of each face is a simple finite polygon.

The *dual* $G^* = (V^*, E^*, F^*)$ of a (finite or infinite) planar graph G is defined as the abstract planar graph whose vertices V^* correspond to the faces F of G ($V^* = F$), and whose edges E^* are in one-to-one correspondence with the edges E of G, as follows: if two faces U and W share an edge e, then the dual vertices U^* and W^* are connected by the dual edge e^*. *We note that even when the underlying graph $G = (V, E, F)$ of a framework (G, p) is a planar graph, the particular placement p of the framework may have crossings: we still may refer to the realization of a face, although it may be a self-intersecting polygon.*

[1]Note that we use *planar* for the graph, as is customary in graph theory, and *non-crossing* for the framework. Our use of *planar framework* is customary in rigidity theory, and refers to a placement in the *plane*.

An edge $\{u, w\}$ of a planar non-crossing framework induces a segment $[p(u), p(w)]$, and it belongs to the boundary of *exactly two faces*, say U and W. In the dual graph $G^* = (F, E)$ these two faces U and W represent two vertices connected by the unoriented edge $\{U, W\}$ *dual* to $\{u, w\}$. Later on, we will need to match an *oriented* edge in the primal graph $G = (V, E)$ with an orientation of its dual edge in $G^* = (F, E)$. We use the following convention. The oriented edge segment $[p(u), p(w)]$ gives opposite senses for going around face U and face W, say counter-clockwise around W and clockwise around U. Then the matching orientation of the edge in the dual graph G^* is from U to W. This matching orientation on a pair of dual edges gives a well-formed double pair $((u, w), (U, W))$, or simply a *tetrad* (u, w, U, W). In computational geometry, these tetrads are implicit in the quad-edge data structure used for representing general surfaces [16]. Reversing orientation on the edge gives the tetrad (w, u, W, U). Below, we will refer to cycles (called *face-cycles*) of oriented edges in the dual graph $G^* = (F, E)$: the orientation rule described above gives an unambiguous correspondence with *oriented edges* (u, v) in the primal graph G and their corresponding *edge vectors* $p(v) - p(u)$ in a geometric placement p of G.

Stressed frameworks. An *equilibrium stress* or, shortly, a *stress* on a planar (finite or infinite, possibly crossing) framework[2] is an assignment $s : E \to \mathbb{R}$ of scalar values $\{s_e\}_{e \in E}$ to the edges E of G in such a way that the edge vectors incident to each vertex $v \in V$, scaled by their corresponding stresses, are in equilibrium, i.e. sum up to zero:

$$\sum_{e=\{u,v\} \in E} s_e(p(v) - p(u)) = 0, \quad \text{for fixed } u \in V \qquad (1)$$

When all s_e are zero, the stress s is *trivial*; when all $s_e \neq 0$, the stress is called *nowhere zero*. The space of all equilibrium stresses of a framework is a vector space, so if a framework has a non-trivial stress, then it is not unique; in particular, any rescaling of it is also a stress.

Lifting. A lifting of the planar framework (G, p) is a continuous function $H : \mathbb{R}^2 \to \mathbb{R}$ whose restriction to any face is an affine function. The lifting assigns a height $H(q)$ to each point q in $\mathbb{R}^2$ (seen as the plane $z = 0$ in $\mathbb{R}^3$), in such a way that the lifted polygonal faces are flat (all cycle vertices lie in the same plane) and connect continuously along the edge segments. The height function is completely determined by the values $H(p(v))$ at the vertices of the framework, and its graph appears as a polyhedral surface or *terrain* over the face-tiling in the reference plane. A lifting is *trivial* if all its faces lie in the same plane, and *strict* if no two adjacent faces are coplanar.

We now move on to the correspondences involved in Maxwell's theorem.

Stress associated to lifting. Let H be a lifting of a framework (G, p). With usual dot product notation, the expression of H restricted to a face U takes the form $H(q) = \nu_U \cdot q + C_U$, for $q \in U \subset \mathbb{R}^2$, where $\nu_U \in \mathbb{R}^2$ is the projection on the reference plane of the *normal* to face U and $C_U \in \mathbb{R}$.

The vectors and constants $H \equiv (\nu_U, C_U)_{U \in F}$ are subject to the compatibility conditions on edges $\{u, v\}$ shared by pairs

[2]Also called a *self-stress* in the rigidity theory literature.

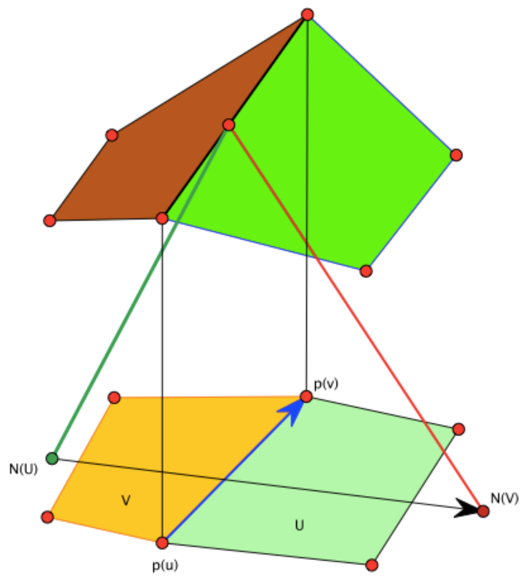


Figure 6: The normals to two adjacent lifted faces induce the dual orthogonal edge.

of adjacent faces $\{U, V\}$:

$$\nu_U \cdot p(u) + C_U = \nu_V \cdot p(u) + C_V, \quad \nu_U \cdot p(v) + C_U = \nu_V \cdot p(v) + C_V \tag{2}$$

We infer that the vector $\nu_V - \nu_U$ is orthogonal to the edge vector $p(v) - p(u)$. Fig. 6 illustrates the relationship.

Given a tetrad (u, v, U, V) of dual edges, and using the notation $(x, y)^\perp = (-y, x)$ for the clockwise rotation with $\pi/2$ of a vector $(x, y) \in \mathbb{R}^2$, we define the *stress* on the edge $\{u, v\} \in E$, associated to the lifting H, as being the proportionality factor s_{uv} given by

$$\nu_V - \nu_U = s_{uv}(p(v) - p(u))^\perp \tag{3}$$

Since the sum involves the vectors around a closed polygon (the face-cycle around a vertex), the equilibrium condition (1) is satisfied. We have shown:

PROPOSITION 1. *For any lifting H of a planar non-crossing framework (G, p), there exists a canonically associated stress on the framework.*

This correspondence between liftings and stresses is essentially the one given by Maxwell, who formulated it through the following geometric construction. The normal direction to the planar
region corresponding to a face $U \in F$ in the lifted terrain is given by $N_U = (\nu_U, -1) \in \mathbb{R}^3$, $U \in F$. When all these normal vectors are taken through the point $(0, 0, 1)$, they intersect the reference plane $z = 0$ in the system of points $\{\nu_U \in \mathbb{R}^2\}_{U \in F}$. The classical "theorem of the three perpendiculars" implies the orthogonality observed above $(\nu_V - \nu_U) \cdot (p(v) - p(u)) = 0$.

Reciprocal diagram. A framework (G^*, p^*) associated to the dual graph G^* of a planar framework (G, p) is called a *reciprocal diagram* if the corresponding primal-dual edges are perpendicular. If in the previous construction we join the points $\{\nu_U \in \mathbb{R}^2 \mid U \in F\}$ by edges dual to the primal ones, we obtain a reciprocal diagram associated to the lifting H. We note that it is possible for several vertices ν_U to coincide, and this happens precisely when the planar regions in the lifting have identical normal directions. An extreme case arises for liftings with globally affine functions H. They give a planar (trivial) terrain over the reference plane, have constant ν_U and induce the trivial stress $\{s_e\}_{e \in E} = 0$.

From stresses to liftings. The direction from stresses to Maxwell liftings requires more work.

PROPOSITION 2. *Let $s = (s_e)_{e \in E}$ be an equilibrium stress for the framework (G, p). Then there exists a lifting H which induces s, determined up to addition of a global affine function.*

Proof: We have to find a set of parameters (ν_U, C_U), indexed by faces and satisfying the face compatibility and orthogonality conditions (2 and 3) in terms of the given placement p. Let us choose an initial face U_0 with an arbitrary lifting $(\nu_{U_0}, C_{U_0}) = (\nu_0, C_0)$. We show that once this initial choice has been made, the lifting is then uniquely determined by the stress values and p.

We solve the linear system (2) in a step-by-step manner, progressing from face to adjacent face, starting at U_0. We consider a *path* through adjacent faces labeled $U_0, U_1, ..., U_n$, with corresponding liftings (ν_i, C_i) and successive tetrads (p_i, q_i, U_i, U_{i+1}). The common edge between faces U_i and U_{i+1} is $[p_i, q_i]$, with the proper orientation. The given stress on this edge is denoted here by s_i. The previous relationship between the stress on an edge and its pair of reciprocal edge vectors implies:

$$C_{k+1} = C_k - (\nu_{k+1} - \nu_k) \cdot p_k = C_k - \sum_{i=0}^{k} s_i (q_i - p_i)^\perp \cdot p_i$$

and

$$\nu_{k+1} = \nu_k + \sum_{i=0}^{k} s_i (q_i - p_i)^\perp \tag{4}$$

Using the identity $(q_i - p_i)^\perp \cdot p_i = det(q_i \; p_i) = |q_i \; p_i|$, the expression of the height function becomes:

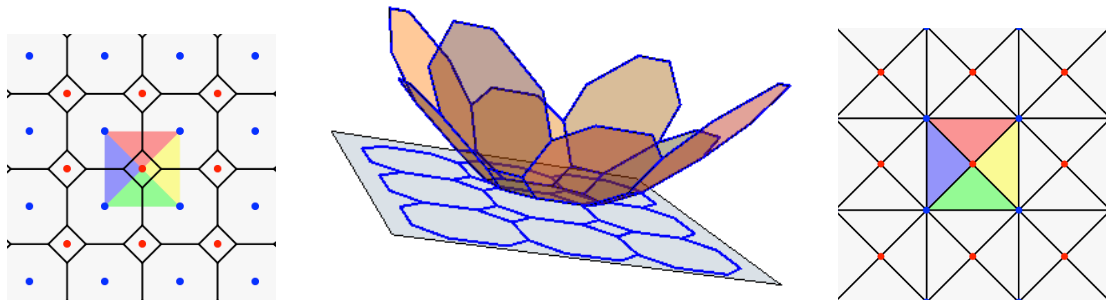
$$H(p) = \nu_n \cdot p + C_n =$$

$$(\nu_0 + \sum_{i=0}^{n-1} s_i (q_i - p_i)^\perp) \cdot p + (C_0 - \sum_{i=0}^{n-1} s_i |q_i \; p_i|) \tag{5}$$

It remains to check that the expression (5) is independent of the face-path chosen from U_0 to U_n. To verify this property, we have to check that the following sums vanish for any face-cycle:

$$\sum_{face-cycle} s_i (q_i - p_i) = 0 \quad and \quad \sum_{face-cycle} s_i |q_i \; p_i| = 0 \tag{6}$$

It suffices to verify these relations over face-cycles corresponding to simple topological loops. In this case, Jordan's simple curve theorem gives a set of vertices inside the loop. When we take the sum over these vertices of the identities (1) and



(a) (b) (c)

Figure 7: (a) A Delauney framework (in black) corresponding to a periodic set with four site orbits under maximal translational symmetry. Dual vertices are shown as colored centers of the faces. Primal-dual edge pairs are orthogonal. (b) The equilibrium stress induced by the reciprocal diagram (c) has a non-periodic lifting to the paraboloid $z = x^2 + y^2$. (c) The reciprocal diagram of the periodic graph (a) is the Voronoi diagram of the periodic sites.

$$\sum_{\{u,v\}\in E} s_{uv}|p(v)\ p(u)| =$$

$$|\sum_{\{u,v\}\in E} s_{uv}(p(v)-p(u))\ p(u)| = 0, \quad \text{for fixed } u,$$

the stress condition (1) implies that terms cancel in pairs for adjacent vertices and leave exactly the desired identities (6). Since the initial choice (ν_0, C_0) was arbitrary, the lifting H is determined only up to a global affine function. ☐

Mountain-valley edge liftings and stress signs. A non-flat edge in the lifting is a *mountain* edge if the terrain is concave in its neighborhood, and a *valley* otherwise. The correspondence between stresses and liftings can be further refined by this well-known property [10] (Fig. 1):

PROPOSITION 3. *The Maxwell correspondence between stressed graphs and liftings takes planar edges with a negative stress to mountain edges in the 3D lifting, those with positive stress to valley edges and those with zero stress to flat lifted edges.*

We conclude this section with a noteworthy example, which will be used to illustrate the critical distinction between equilibrium stress and periodic stress.

EXAMPLE 1 (**Periodic Voronoi/Delauney pair**). The classical Voronoi-Delauney duality, applied to infinite point sets, in particular to periodic ones, yields a dual pair of frameworks whose corresponding dual edges are orthogonal. The classical lifting on the paraboloid shows that these frameworks support an equilibrium stress. See Fig. 7.

3. EQUILIBRIUM AND PERIODIC STRESS OF A PERIODIC FRAMEWORK

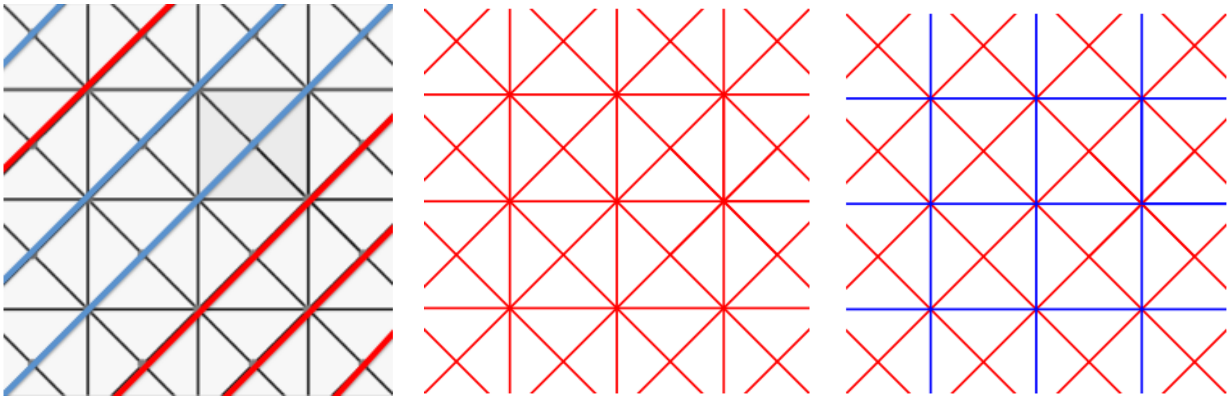
We extend now the relationships obtained in the previous section to infinite *periodic* frameworks, as defined in [2, 3] and specialized to connected, non-crossing frameworks in the plane.

Periodic frameworks. A periodic framework, denoted as (G, Γ, p, π), is given by an infinite graph G, a periodicity group Γ acting on G, and geometric realizations p and π of these two objects. The graph $G = (V, E)$ is simple (has no multi-edges and no loops) and connected, with an infinite set of vertices V and undirected edges E. The *periodicity group* $\Gamma \subset Aut(G)$ is a free Abelian group of rank two acting on G without fixed points. We consider only the case when the quotient multigraph G/Γ (which may have loops and multiple edges) is *finite*, and use $n = |V/\Gamma|$ and $m = |E/\Gamma|$ to denote the number of vertex and edge orbits. The function $p : V \to \mathbb{R}^2$ gives a specific placement of the vertices as points in the plane, in such a way that any two vertices joined by an edge in E are mapped to distinct points. The *injective group morphism* $\pi : \Gamma \to \mathcal{T}(\mathbb{R}^2)$ gives a faithful representation of Γ by a *lattice of translations* $\pi(\Gamma) = \Lambda$ of rank two in the group of planar translations $\mathcal{T}(\mathbb{R}^2) \equiv \mathbb{R}^2$. The placement is *periodic* in the obvious sense that the abstract action of the periodicity group Γ is replicated by the action of the periodicity lattice $\Lambda = \pi(\Gamma)$ on the placed vertices: $p(\gamma v) = \pi(\gamma)(p(v))$, for all $\gamma \in \Gamma, v \in V$.

Non-crossing periodic frameworks. We consider now periodic frameworks which are, as infinite frameworks, non-crossing. We have an underlying periodic planar graph $G = ((V, E, F), \Gamma)$ on which the periodicity group Γ acts. The *dual periodic planar graph* $G^* = ((V^*, E^*, F^*), \Gamma)$ is obtained from the abstract dual of the infinite graph $G = (V, E, F)$, defined as above. Since the periodicity group Γ acts on it in the same manner as it acts on the primal graph G, G^* is itself a periodic planar graph. If we denote by $n^* = card(F/\Gamma)$ the number of face orbits under Γ, then Euler's formula for the torus $\mathbb{R}^2/\Lambda \supset G/\Gamma$ gives the relation: $n - m + n^* = 0$, that is $n + n^* = m$.

Equilibrium stress on periodic frameworks. A stress s of the planar periodic framework (G, p, Γ, π) is called a Γ-*invariant equilibrium stress* if it is invariant on edge orbits E/Γ. A Γ-invariant equilibrium stress can be calculated by solving a finite linear system of equations of type (1), where the unknowns are the stresses for the edge representatives in E/Γ and the equations correspond to equilibrium conditions for the vertex representatives in V/Γ.

Periodic liftings. A lifting H for the planar periodic framework (G, Γ, p, π) is called *periodic* if it is Γ-invariant, i.e. if



(a) (b) (c)

Figure 8: The Voronoi diagram from Fig. 7(c) is a periodic framework supporting different types of stresses. (a) (Not Γ-invariant) Each of the "aligned" infinite paths supports a one-dimensional stress (equal on each edge of the path and illustrated here with the oblique colored diagonals). The stress values can be independently chosen on each diagonal path. Thus they yield equilibrium stresses, such as the one depicted here, which are not Γ-invariant. (b) (Non-periodic Γ-invariant) A Γ-invariant stress assigns the same stress value on all edges in an edge orbit. Illustrated here is a stress where all stress orbits have the same sign; this cannot be a periodic stress. (c) A periodic stress, as the one shown here, must have both positive and negative stresses on edge orbits.

$H(q+\lambda) = H(q)$, for all $q \in \mathbb{R}^2$ and $\lambda \in \Lambda = \pi(\Gamma)$, where translation by periods λ has been written additively.

Not all liftings of periodic frameworks are Γ-invariant. An example arises from the classical lifting of points (x, y) to points (x, y, x^2+y^2) on a paraboloid, which lifts the Voronoi diagram (and its dual Delauney tessellation) to a polyhedral surface tangent to, resp. inscribed into the paraboloid. An important observation is that *Γ-invariant equilibrium stresses do not necessarily yield periodic liftings.* Indeed, considerations of symmetry show that the periodic Voronoi diagram in Fig. 7 has a Γ-invariant equilibrium stress, but the induced lifting onto the paraboloid is obviously not periodic.

Since Γ-invariant stresses may not always correspond to Γ-invariant liftings, additional properties are needed to characterize stresses induced by Γ-invariant liftings. We proceed now to find them.

Stress induced by a periodic lifting. When expressed as the system $H \equiv (\nu_U, C_U)_{U\in F}$, a periodic lifting has constant coefficients ν_U on Γ-orbits of faces. Thus, the corresponding placement of the dual graph (i.e. the reciprocal diagram) has at most $n^* = card(F/\Gamma)$ distinct vertices.

For a closer investigation of the associated stress, we introduce the following notational conventions. A face-path from a face U to a face V will be indicated by $U \to V$, in particular, a face-path from U to its translate $U+\lambda$ will be indicated by $U \to U+\lambda$. *Sums over face-paths or face-cycles are assumed to be written according to the orientation rule given through tetrads.*

With this convention, we rewrite the relations (4) obtained in Section 2 as:

$$\nu_V = \nu_U + \sum_{U\to V} s_i (q_i - p_i)^{\perp}$$

and

$$C_V = C_U - \sum_{U\to V} s_i |q_i \; p_i| \tag{7}$$

The transition formula (5) applied to a periodic lifting H gives the identity

$$\nu_{U+\lambda} \cdot (p+\lambda) + C_{U+\lambda} =$$

$$(\nu_U + \sum_{U\to U+\lambda} s_i (q_i - p_i)^{\perp}) \cdot (p+\lambda) + C_{U+\lambda} = \nu_U \cdot p + C_U$$

for all $p \in U$. Thus, the stress s must satisfy the following two conditions:

$$\sum_{U\to U+\lambda} s_i (q_i - p_i) = 0 \tag{8}$$

$$\sum_{U\to U+\lambda} s_i |q_i \; p_i| = C_U - C_{U+\lambda} = \nu_U \cdot \lambda \tag{9}$$

We summarize these observations as:

Proposition 4. *If the planar periodic framework (G, Γ, p, π) has a periodic lifting $H \equiv (\nu_U, C_U)_{U\in F}$, then the associated Γ-invariant equilibrium stress s satisfies conditions (8) and (9) for any face U and period vector $\lambda \in \Lambda = \pi(\Gamma)$.*

From constrained stress to periodic lifting. The two conditions (8) and (9) are also sufficient for determining a periodic lifting (the proof is given in the full paper):

Proposition 5. *Let $s = (s_e)_{e\in E}$ be a Γ-invariant equilibrium stress for the planar non-crossing periodic framework (G, Γ, p, π). If, for some face U_0 and generators λ_1, λ_2 of the* *period lattice $\Lambda = \pi(\Gamma)$, the stress s satisfies the additional conditions that:*

$$\sum_{U_0 \to U_0+\lambda_j} s_i(q_i - p_i) = 0, \quad j = 1, 2 \tag{10}$$

then the lifting $H \equiv (\nu_U, C_U)$ defined, for all $\lambda \in \Lambda$, by:

$$\nu_U \cdot \lambda = \sum_{U \to U+\lambda} s_i |q_i \; p_i| = C_U - C_{U+\lambda}$$

is determined up to a choice of constant $C_0 = C_{U_0}$, and is a periodic lifting inducing s.

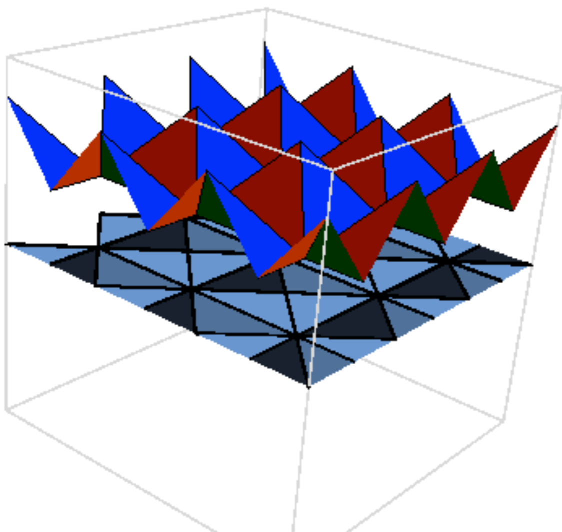


Figure 9: A periodic 3D lifting for the stressed framework in Figure 8(c).

In the next subsection we prove that this type of stress is precisely the periodic stress involved in the deformation theory of periodic frameworks introduced in [2].

4. PERIODIC MOTIONS AND STRESSES

We have arrived at one of the most important aspects of this paper, which brings in the connection with the infinitesimal rigidity of a periodic framework. The application to periodic pseudo-triangulations and expansive mechanisms presented in the next section relies on this correspondence.

Periodic deformations. A planar framework (G, Γ, p, π) was defined by a placement of vertices $p : V \to \mathbb{R}^2$ and a faithful representation $\pi : \Gamma \to \mathcal{T}(\mathbb{R}^2)$ of the periodicity group by a rank two lattice of translations $\Lambda = \pi(\Gamma)$, with the necessary compatibility relation. In the framework, the edges of the graph are now seen as segments of fixed length, forming what is called in rigidity theory a *bar-and-joint structure.* According to our formulation of a periodic deformation theory, introduced in [2] and pursued in [3, 5], a *periodic bar-and-joint framework* is said to be *periodically flexible* if there exists a continuous family, parametrized by time t, of placements $p_t : V \to \mathbb{R}^2$ with $p_0 = p$, which satisfies two conditions: (a) it maintain the lengths of all the edges $e \in E$, and (b) it maintains periodicity under Γ, via faithful representations $\pi_t : \Gamma \to \mathcal{T}(\mathbb{R}^2)$ which *may change with t and give an associated variation of the periodicity lattice $\Lambda_t = \pi_t(\Gamma)$.*

To represent π_t we first choose two generators for the periodicity lattice Γ. The corresponding lattice generators $\lambda_1(t)$ and $\lambda_2(t)$ at time t may be viewed as the columns of a non-singular 2×2 matrix denoted, for simplicity, with the same symbol $\Lambda_t \in GL(2)$. The infinitesimal deformations of the placement (p_t, π_t) are described using a complete set of n vertex representatives for V/Γ, i.e. the vertex positions are parametrized by $(\mathbb{R}^2)^n$. The m representatives for edges mod Γ are then expressed using the vertex parameters and the periodicity matrix Λ. An edge representative β originates in one of the chosen vertex representatives $i = i(\beta)$ and ends at some other vertex representative $j = j(\beta)$ *plus some period* Λc_β, where c_β is a column vector with two integer entries. The edge vectors e_β, $\beta \in E/\Gamma$ thus have the form:

$$e_\beta = (x_j + \Lambda c_\beta) - x_i, \quad \beta \in E/\Gamma \tag{11}$$

By taking the squared length of the m edge representatives, we obtain a map $(\mathbb{R}^2)^n \times GL(2) \to \mathbb{R}^m$. The differential of this map at the point of $(\mathbb{R}^2)^n \times GL(2) \subset \mathbb{R}^{2n+4}$ corresponding to the framework (p, π), seen as a matrix with m rows and $2n + 4$ columns, is called the *rigidity matrix* $\mathbf{R} = \mathbf{R}(G, \Gamma, p, \pi)$ of the framework. Denoting by e^t_β the transpose of the column edge vector e_β and using an obvious grouping convention for the columns corresponding to individual vertices, the row corresponding to the edge β described above is:

$$(0...0 \; -e^t_\beta \; 0...0 \; e^t_\beta \; 0...0 \; c^1_\beta e^t_\beta \; c^2_\beta e^t_\beta) \tag{12}$$

The vector space of *infinitesimal periodic motions* of the given framework (G, Γ, p, π) can now be described as the *kernel* of the rigidity matrix $\mathbf{R}$ and the vector space of *periodic stresses* can be described as the *kernel* of the transpose $\mathbf{R}^t$. A stress described on the m representatives for E/Γ is extended by periodicity to all edges.

Thus, non-trivial periodic stresses express linear dependencies between the rows of the rigidity matrix $\mathbf{R}$. Grouping these dependencies over groups of columns corresponding to vertex representatives, we obtain immediately that a periodic stress satisfies conditions (1) and thus is necessarily a Γ-invariant equilibrium stress. However, there are two *additional* vector conditions imposed by the columns corresponding to the infinitesimal variation of the periods.

This sets the stage for a comparison of the periodic stresses reviewed here and the stresses induced by periodic liftings. For clarity we restate the definition:

Definition 1. [2] A periodic stress for the framework (G, Γ, p, π) is a stress induced from an element in the kernel of the transposed rigidity matrix $\mathbf{R}^t$, that is, a Γ-invariant equilibrium stress s satisfying the additional conditions:

$$\sum_{\beta \in E/\Gamma} s_\beta c^j_\beta e_\beta = 0 \tag{13}$$

with integer coefficients c^j_β, $j = 1, 2$, as given in the edge description (11).

Our next goal is to relate the conditions (10) and (13). For this, we first show the *persistence of periodic stresses under relaxation of periodicity* from Γ to a subgroup of *finite index* $\tilde{\Gamma} \subset \Gamma$.

Proposition 6. *Let $s = (s_\beta)_{\beta \in E}$ be a periodic stress for the periodic framework (G, Γ, p, π). Let $\tilde{\Gamma} \subset \Gamma$ be a subgroup of finite index. Then s remains a periodic stress for the framework with relaxed periodicity $(G, \tilde{\Gamma}, p, \pi|_{\tilde{\Gamma}})$. Moreover, if a Γ-invariant equilibrium stress is periodic for a relaxed periodicity $\tilde{\Gamma} \subset \Gamma$, it is already periodic for Γ.*

The proof appears in the full paper. As a consequence we obtain that, upon relaxation of periodicity, *the dimension of the space of periodic stresses can only go up or stay the same.*

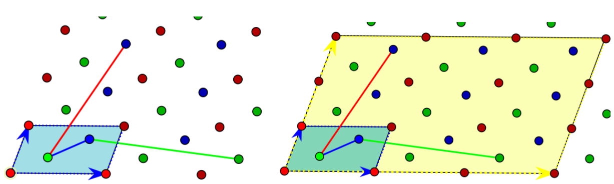
We recall (from [2], page 2641) the relation $\sigma - \delta = m - 2n - 4$ connecting periodic stresses and infinitesimal deformations, where σ denotes the dimension of the space of periodic stresses and δ is the dimension of the space of infinitesimal periodic deformations. Subtracting the trivial infinitesimal deformations induced by infinitesimal isometries, we obtain:

$$\sigma = \phi - 1 + (m - 2n) \tag{14}$$

where ϕ denotes the dimension of the space of infinitesimal flexes $\phi = \delta - 3$. This formula is relevant for evaluating behavior under relaxations, with σ and ϕ non-decreasing and the term $(m - 2n)$ multiplied by the index of relaxation ρ.

The last ingredient needed for the proof of our Main Theorem is Lemma 7 below, whose detailed formulation appears in the full paper. It gives a more transparent interpretation for the conditions (13) satisfied by periodic stresses, in terms of a sufficiently large relaxation of periodicity. We denote by P_Γ a fundamental parallelogram for the periodicity group Γ.

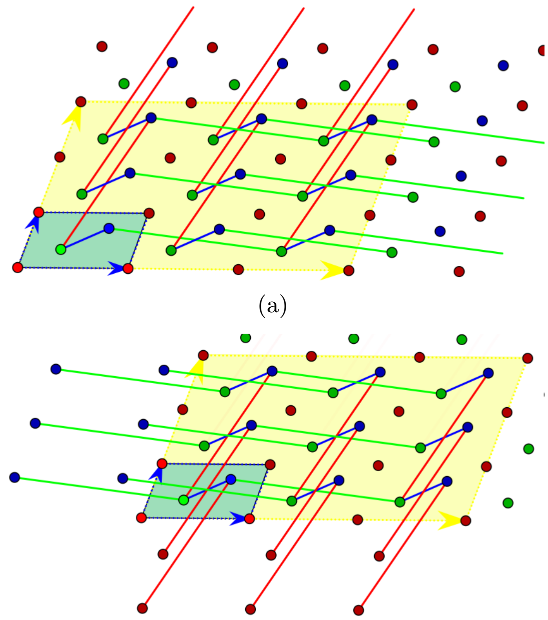
Lemma 7. *For sufficiently relaxed periodicity $\tilde{\Gamma}$, one can find a complete set of edge representatives which are either inside $P_{\tilde{\Gamma}}$ or cross its border to a neighboring parallelogram over two chosen sides.*



(a) (b)

Figure 10: Fundamental parallelograms P_Γ and $P_{\tilde{\Gamma}}$ for the proof of Lemma 7. (a) Edge representatives for the three edge orbits in the periodic graph (G, Γ). (b) The relaxation $\tilde{\Gamma}$ of the lattice. The edge representatives for $(G, \tilde{\Gamma})$ are shown in Fig. 11.

The idea for obtaining edge representatives for E/Γ is to take all edges contained in the fundamental parallelogram P_Γ and then to select the remaining representatives from edges originating in P_Γ and crossing the boundary of P_Γ, as in Fig. 10(a). We then find a large enough integer r such that the dilated parallelogram rP_Γ, adequately translated over P_Γ, would contain inside all edges originating in P_Γ, as in Fig. 10(b). The edge representatives of the relaxed periodic graph $(G, \tilde{\Gamma})$ are either inside the new fundamental parallelogram $P_{\tilde{\Gamma}}$ or cross over to one of the neighboring parallelograms. It is always possible to select the edge representatives of the second type to cross the two chosen generators of the lattice. Fig. 11(a) illustrates the full set of edge representatives for the relaxed periodic graph in Fig. 10(b), with the final choices crossing the selected lines shown in Fig. 11(b).



(a)

(b)

Figure 11: Illustration of the construction used in the proof of Lemma 7: standardizing edge representatives, chosen to cross the two generators of the relaxed periodicity lattice. The edge colors indicate the orbits relative to the original lattice Γ, not $\tilde{\Gamma}$.

We are now ready for the proof of the correspondence between periodic liftings and stresses.

Main Theorem *Let (G, Γ, p, π) be a planar non-crossing periodic framework. A stress induced by a periodic lifting is a periodic stress and conversely, any periodic stress is induced by a periodic lifting, determined up to an arbitrary additive constant. The correspondence relates the stress signs to the mountain/valley types of the lifted edges.*

Proof. We use Prop. 6 and the setting described in Lemma 7 obtained after an adequate relaxation of periodicity $\tilde{\Gamma} \subset \Gamma$ with generators related by $\tilde{\lambda}_j = r_j \lambda_j, j = 1, 2$. We first observe that, for periodicity $\tilde{\Gamma}$ the stated correspondence between periodic liftings and periodic stresses becomes obvious, since conditions (13) and (10) ask exactly the same thing: that the stress-weighted sums of edges involved along $U \to U + \tilde{\lambda}_j, j = 1, 2$ be zero. The case of full periodicity Γ now follows from Proposition 6 and the corresponding fact that a Γ-invariant lifting which is $\tilde{\Gamma}$-periodic for some relaxation $\tilde{\Gamma} \subset \Gamma$, must be already Γ-periodic, as immediately seen from conditions (10). The sign relationship follows from Prop. 3. □

5. PERIODIC POINTED PSEUDO-TRIANGULATIONS

A *pseudo-triangle* is a simple closed planar polygon with exactly three internal angles smaller than π. A set of vectors with a common origin is *pointed* if they lie in some open half-plane determined by a line through their origin. A planar non-crossing periodic framework (G, Γ, p, π) is a *periodic pointed pseudo-triangulation* when all faces are pseudo-triangles and the framework is pointed at every vertex. As in the finite case, pointedness at every vertex is essential. Pseudo-triangular faces mark the 'saturated' stage where no more edge orbits can be inserted without violating non-crossing or pointedness. An illustration for $n = 3$ is given in Fig. 12. We show that periodic pointed pseudo-triangulations, viewed as bar-and-joint mechanisms, satisfy two remarkable rigidity-theoretic properties: they have the right number of edges to be flexible mechanisms with exactly one degree of freedom (in the finite case [28, 29], the flexible mechanisms were obtained after removing a convex hull edge), and they encounter no singularities in their deformation for as long as they remain pseudo-triangulations.

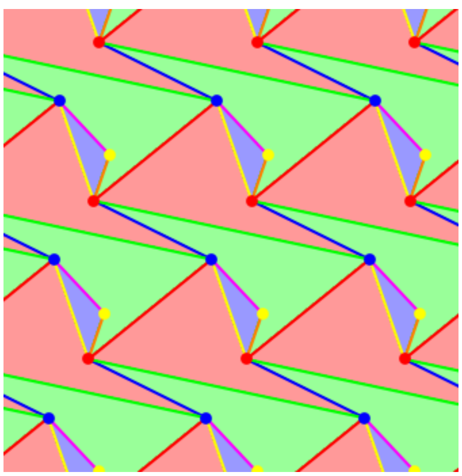


Figure 12: A periodic pseudo-triangulation with $(n, m, n^*) = (3, 6, 3)$.

PROPOSITION 8. *A periodic pseudo-triangulation has $m = 2n$, that is, the number of edge orbits $m = card(E/\Gamma)$ is twice the number of vertex orbits $n = card(V/\Gamma)$.*

PROPOSITION 9. *A periodic pseudo-triangulation cannot have nontrivial periodic stresses. The local deformation space is therefore smooth and one-dimensional and continues to be so as long as the deformed framework remains a pseudo-triangulation. The same statement holds true for any relaxation of periodicity $\tilde{\Gamma} \subset \Gamma$ of finite index.*

Finally, combining these results with our Main theorem, we obtain:

THEOREM 10. *Let (G, Γ, p, π) be a planar periodic pseudo-triangulation. Then the framework has a one-parameter periodic deformation, which is expansive for as long as it remains a pseudo-triangulation.*

6. APPLICATIONS

Ultrarigidity. By our results from [2], the quotient graph of a generic minimally rigid periodic framework has $m = 2n + 1$ edges, and must satisfy a simple sparsity condition, which is easy to verify for periodic pointed pseudo-triangulations. Hence, by adding one edge-orbit, we can turn periodic pseudo-triangulations into minimally rigid frameworks. Morover, these frameworks remain infinitesimally rigid *for any relaxation of periodicity.* Thus, periodic pseudo-triangulations and insertion choices provide endless examples of ultrarigid frameworks.

Periodic pseudo-triangulations are auxetic. We have recently [6] formulated a definition of *auxetic behavior* and proved that expansive mechanisms exhibit auxetic behavior. Our geometric approach relies on the evolution of the periodicity lattice. Let us consider a differentiable one-parameter deformation $(G, \Gamma, p_\tau, \pi_\tau), \tau \in (-\epsilon, \epsilon)$ of a periodic framework. After choosing an independent set of generators for Γ, the image $\pi_\tau(\Gamma)$ is completely described via the $d \times d$ matrix Λ_τ with column vectors given by the images of the generators under π_τ. The associated Gram matrix will be

$$\omega(\tau) = \Lambda_\tau^t \Lambda_\tau.$$

A deformation path $(G, \Gamma, p_\tau, \pi_\tau), \tau \in (-\epsilon, \epsilon)$ is auxetic if and only if the curve of Gram matrices $\omega(\tau)$ defined above has all its tangents in the cone of positive semidefinite symmetric $d \times d$ matrices.

Since expansive implies auxetic, *periodic pointed pseudo-triangulations provide an infinite family of auxetic frameworks.* As we remarked in the introduction, only a few sporadic examples were previously known, and their auxetic properties were based on empirical observations rather than proven mathematically.

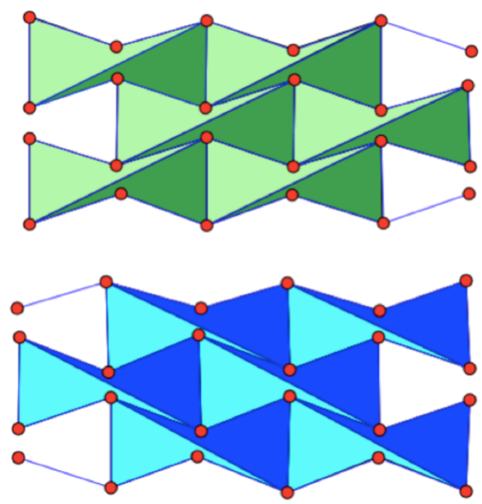


Figure 13: Two possible refinements to periodic pseudo-triangulations of the 'reentrant' structure of hexagons in Figure 5.

Kinematics of periodic expansive frameworks. Planar periodic frameworks which allow expansive one-parameter deformations can be described in terms of pseudo-triangulations.

Example 2. **(The "reentrant honeycomb")** *The framework in Fig. 5 has two degrees of freedom and not all of its deformation paths are auxetic. The expansive deformations can be explained in terms of the two possible refinements to periodic pseudo-triangulations shown in Fig. 13.*

A complete characterization of those periodic frameworks which allow expansive trajectories, together with a systematic approach for generating expansive motions is presented in [8].

In conclusion, we anticipate that our periodic version of Maxwell's Theorem and the expansive nature of periodic pseudo-triangulations will find, like their finite counterparts, further applications in discrete and computational geometry. In the larger scientific context, applications are expected in new materials and mechanism design.

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