On Highly Regular Embeddings (Extended Abstract)

Pavle V. M. Blagojević, Wolfgang Lück, and Günter M. Ziegler

Abstract A continuous map $\mathbb{R}^d \to \mathbb{R}^N$ is *k-regular* if it maps any k pairwise distinct points to k linearly independent vectors. Our main result on k-regular maps is the following lower bound for the existence of such maps between Euclidean spaces, in which $\alpha(k)$ denotes the number of ones in the dyadic expansion of k:
For $d \geq 1$ and $k \geq 1$ there is no k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ for $N < d(k-\alpha(k)) + \alpha(k)$.
This reproduces a result of Chisholm from 1979 for the case of d being a power of 2; for the other values of d our bounds are in general better than Karasev's (2010), who had only recently gone beyond Chisholm's special case. In particular, our lower bound turns out to be tight for $k \leq 3$.
The framework of Cohen & Handel (1979) relates the existence of a k-regular map to the existence of a specific inverse of an appropriate vector bundle. Thus non-existence of regular maps into $\mathbb{R}^N$ for small N follows from the non-vanising of specific dual Stiefel–Whitney classes. This we prove using the general Borsuk–Ulam–Bourgin–Yang theorem combined with a key observation by Hung (1990) about the cohomology algebras of configuration spaces.
Our study produces similar topological lower bound results also for the existence of *ℓ-skew embeddings* $\mathbb{R}^d \to \mathbb{R}^N$ for which we require that the images of the tangent spaces of any ℓ distinct points are skew affine subspaces. This extends work by Ghomi & Tabachnikov (2008) for $\ell = 2$.

The details for this work are provided in our paper *On highly regular embeddings*, preprint, May 2013, 19 pages; http://arxiv.org/abs/1305.7483.

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1 Introduction

Let $d \geq 1$ and $k \geq 1$. A map $\mathbb{R}^d \to \mathbb{R}^N$ is *k-regular* if it maps any k pairwise distinct points to k linearly independent vectors. Such a map only exists if N is large enough. How large does N have to be?

The study of the existence of k-regular maps was initiated by Borsuk [5] in 1957 and latter attracted additional attention due to its connection to approximation theory, via the Haar–Kolmogorov–Rubinstein theorem (see [7]). The problem and its extensions were extensively studied by Chisholm, F. Cohen, Handel, and others in the 1970's and 1980's [6, 7, 9, 10], and then again by Handel and Vassiliev in the 1990s [11, 14, 15].

Some basic examples and results are as follows, where $N(d,k)$ denotes the smallest dimension for which a k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ exists:

- $N(d,k) \geq k$ is trivial.
- $N(d,1) = 1$.
- $N(d,2) = d+1$.
- $N(1,k) = k$ follows from the existence of the real *moment curve*
$$\gamma : \mathbb{R} \to \mathbb{R}^k, \quad t \mapsto (1,t,\dots,t^{k-1}).$$
- $N(2,k) \leq 2k-1$ follows from the *complex moment curve*
$$\gamma_{\mathbb{C}} : \mathbb{C} \to \mathbb{R} \times \mathbb{C}^{k-1}, \quad t \mapsto (1,t,\dots,t^{k-1}).$$
- $N(d,3) \leq d+2$ is obtained from embeddings
$$\mathbb{R}^d \hookrightarrow S^d \hookrightarrow \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \times \{1\} \hookrightarrow \mathbb{R}^{d+2}.$$
- $N(d,k) \geq (d+1)\lfloor \frac{k}{2} \rfloor$ was proven by Boltyansky, Ryshkov & Saskin [4].
- $N(d,k) \leq (d+1)k$ may be obtained from a general position smooth embedding, see Boltjanski [3] and Handel [11].

2 Main Result

Our main result on k-regular maps — see [2] for details — is the following lower bound for the existence of such maps between Euclidean spaces:

Theorem 1. *For any $d \geq 1$ and any $k \geq 1$ there is no k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ for*
$$N < d(k-\alpha(k)) + \alpha(k),$$
where $\alpha(k)$ denotes the number of ones in the dyadic expansion of k

This reproduces a result of Chisholm [6] from 1979 for the case when d is a power of 2; for the other values of d our bounds are in general better than Karasev's [13], who had only recently gone beyond Chisholm's special case. In particular, our lower bound turns out to be tight for $k = 3$ (see also Handel [9]).

3 Methods

Any k-regular map $f : \mathbb{R}^d \to \mathbb{R}^N$ yields an $\mathfrak{S}_k$-equivariant map

$$F(\mathbb{R}^d, k) \longrightarrow_{\mathfrak{S}_k} V_k(\mathbb{R}^N)$$

from the configuration space of (ordered) k-tuples of distinct points in $\mathbb{R}^d$ to the Stiefel manifold of ordered k-frames in $\mathbb{R}^N$. Cohen & Handel [7] showed that the existence of such an equivariant map is equivalent to the existence of an $(N-k)$-dimensional inverse of the vector bundle

$$\xi_{d,k} : \mathbb{R}^k \longrightarrow F(\mathbb{R}^d, k) \times_{\mathfrak{S}_k} \mathbb{R}^k \longrightarrow F(\mathbb{R}^d, k)/\mathfrak{S}_k,$$

that is, to the existence of an embedding into a trivial bundle of rank N over the unordered configuration space $F(\mathbb{R}^d, k)/\mathfrak{S}_k$.

Thus if the k-regular map $f : \mathbb{R}^d \to \mathbb{R}^N$ exists, then the dual Stiefel–Whitney class $\overline{w}_{N-k+1}(\xi_{d,k})$ vanishes. Hence Theorem 1 is a consequence of the following, which is our main technical result.

Theorem 2. *For any $d \geq 1$ and $k \geq 1$,*

$$\overline{w}_{(d-1)(k-\alpha(k))}(\xi_{d,k}) \neq 0.$$

Chisholm has proved this for the case $d = 2^e$ in 1979 [6].

Proof. Our proof proceeds in five steps. The first four of them treat the special case $k = 2^m$.

In the first step we pass to the image of the cohomology of $\mathfrak{S}_k$ under the restriction to the cohomology of the subgroup $E_m := (\mathbb{Z}/2)^m$, which is known to be generated by the Dickson invariants $q_{m,i}$ of degree $2^m - 2^i$.

In the second step, we obtain that the pull-back of $q_{m,0}^{d-1}$ along the classifying map is different from 0, by identifying it with $w_{k-1}(\xi_{d,k})^{d-1}$, which is non-zero according to our previous work in [1]. All other monomials in the classes w_i, $i < k-1$, of degree at least $(d-1)(k-1)$, vanish according to a key observation by Hung [12] together with the structure of our model of the configuration space $F(\mathbb{R}^d, n)$ from [1].

In the third step, we proceed by induction on d, based on the general Borsuk–Ulam–Bourgin–Yang theorem.

In the fourth step, we study the monomial expansion of the dual Stiefel–Whitney invariants,

$$\begin{aligned}
\overline{w}_{(d-1)(k-\alpha(k))}(\xi_{d,k}) &= \sum_{\substack{j_1,\dots,j_{k-1}\ge 0\\ j_1+2j_2+\cdots+(k-1)j_{k-1}=(d-1)(k-1)}} \binom{j_1+\cdots+j_{k-1}}{j_1,j_2,\dots,j_{k-1}} w_1^{j_1}\cdots w_{k-1}^{j_{k-1}}\\
&= w_{k-1}^{d-1} + \text{other terms.}
\end{aligned}$$

where Chisholm [6] had exploited that for $d = 2^e$ all the relevant multinomial coefficients vanish mod 2, while we need and get that all the "other terms" are zero.

In the fifth and last step we extend the result to general k.

We refer to [2] for the details.

4 Further work

Our study [2] produces similar topological lower bound results also for the existence of *ℓ-skew embeddings* $\mathbb{R}^d \to \mathbb{R}^N$, for which we require that the images of the tangent spaces of any ℓ distinct points are skew affine subspaces. This extends work by Ghomi & Tabachnikov [8] for $\ell = 2$.

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