Tight and non-tight topological Tverberg type theorems

Günter M. Ziegler

(joint work with Pavle V. M. Blagojević, Benjamin Matschke, and Florian Frick)

A short history of (tight) Tverberg theorems. The history of "Tverberg type" multiple intersection theorems (after the classical convexity results of Helly and Radon, and the non-embeddability results of van Kampen and Flores, etc.) starts with Birch's 1959 paper "On 3N points in a plane" [5], which contained the following three achievements.

Theorem 1: *Any $3N$ points in the plane can be partitioned into N triangles that have a point in common.*

Theorem 1*: *Any $3N-2$ points in the plane can be partitioned into N subsets whose convex hulls have a point in common.*

Conjecture: *Any $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into r subsets whose convex hulls have a point in common.*

We note that Birch's Theorem 1*, as well as his conjecture, which was proved in full by Helge Tverberg in 1964, fifty years ago (see [17]), and thus is now known as Tverberg's theorem [13], are *tight*: This is not only evident from concrete configurations, but also from a general position argument: If $(r-1)(d+1)$ points in $\mathbb{R}^d$ in general position are partitioned into r subsets, then not even their *affine* hulls intersect, as one sees from a codimension count. (See Kalai [9] for far-reaching (conjectured) extensions of this.)

The tightness of the results also means that for a generic point configuration the number of intersection points, known as *Tverberg points*, is finite. With this finiteness it is also very natural to ask for the minimal number of Tverberg r-partitions, which according to Sierksma's conjecture should be $(r-1)!^d$; only a much weaker result is proven [15]; see [10].

In a modern version (a point of view pioneered by in [1]), Tverberg's theorem would be phrased to say that *for $d \geq 1$ and $r \geq 2$, and $N := (r-1)(d+1)$ for any affine map $f : \Delta_N \to \mathbb{R}^d$ the N-dimensional simplex Δ_N contains r points $x_1, \dots, x_r$ that lie in r vertex-disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N whose images coincide: $f(x_1) = \cdots = f(x_r)$.*

In this version, the "topological version" for continuous maps f is natural. This is a breakthrough result by Bárány, Shlosman & Szűcs [4] from 1981, extended from primes r to prime powers by Özaydin [12] in 1987 — which, however, remains a conjecture for the case when r is not a prime power and $d \geq 2$:

The topological Tverberg theorem/conjecture [4] [12]**:** *Let $d \geq 1$ and $r \geq 2$, and $N := (r-1)(d+1)$. For any continuous map $f : \Delta_N \to \mathbb{R}^d$ the N-dimensional simplex Δ_N contains r points $x_1, \dots, x_r$ that lie in r vertex-disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N whose images coincide: $f(x_1) = \cdots = f(x_r)$.*

Again this result is "tight": It fails for *all* general-position maps f if N is replaced by a smaller number.

Colored versions of Tverberg type theorems. In an influential 1989 Computational Geometry paper, Bárány, Füredi, and Lovász observed: *"we need a colored version of Tverberg's theorem."* For their purposes they needed only a very small special case: *Let A, B, C be sets of t points in the plane, then one can find $r = 3$ disjoint triples consisting of one point of each of the three sets such that the convex hulls of the triples have a point in common.* (Here the points in A, B, and C are interpreted as having three different colors.) Bárány et al. proved this for $t = 7$, asserted they also had a proof for $t = 4$, but also noted that they had no counterexample even for $t = 3$. So, in particular, their result was not tight.

The call for a colored version of Tverberg's theorem was seen as a challenge, and attacked immediately. The first answer, by Imre Bárány and David Larman 1990, treated the case of $3r$ points in the plane, with three different colors. In particular, they suggested the following.

The Bárány–Larman colored Tverberg conjecture [3]**:** *Let $d \geq 1$, $r \geq 2$, and $N \geq N(r, d)$ sufficiently large. Assume that $f : \Delta_N \longrightarrow \mathbb{R}^d$ is affine (or at least continuous), where the $N+1$ vertices of Δ_N carry $d+1$ different colors, and every color class has size at least r. Then Δ_N has r disjoint* rainbow faces*, whose images under f intersect.*

Here a *rainbow face* refers to a d-dimensional face of the simplex Δ_N whose $d+1$ vertices carry the $d+1$ different colors. In the case $d = 2$ thus we have at least $3r$ points in the plane, which carry the three different colors. In this situation there should be r rainbow triangles that have a point in common. For $d = 2$ Bárány and Larman proved this, and thus obtained a "sharp" colored version of Birch's Theorem 1. However, for $d > 2$ they did not obtain a finiteness result for $N(r, d)$.

In a "Note added in proof," Bárány and Larman announced that Živaljević and Vrećica had proven the finiteness of $N(r, d)$ — but indeed, they haven't. In their celebrated 1992 paper [16] (see [10]) they established the following result.

Živaljević and Vrećica's colored Tverberg theorem [16]**:** *Let $d \geq 1$ and r a prime. Assume that $f : \Delta_N \longrightarrow \mathbb{R}^d$ is continuous, where the $N+1$ vertices of Δ_N carry $d+1$ different colors, and every color class has size at least $2r-1$. Then Δ_N has r disjoint* rainbow faces*, whose images under f intersect.*

Via Bertrand's postulate, the condition that r is prime may be dropped if we make the color classes bigger, e.g. of size at least $4r - 1$. However, even if in the Bárány–Larman conjecture $N(r, d)$ is taken to be *very* large, then this still does not imply that *all* color classes get large, say larger than $2r - 1$. Thus Živaljević and Vrećica's colored Tverberg theorem does establish the colored Tverberg result suggested by Bárány–Füredi–Lovász, but it does *not* even yield finiteness of $N(r, d)$ in the Bárány–Larman problem.

Furthermore, neither the Bárány–Larman result for $d = 2$ nor the Živaljević–Vrećica version is tight: A tight version should generalize Birch's Theorem 1*, and thus not use more than $3r - 2$ points in the plane!

A first sharp version, with a proof of $N(r, d) = r(d + 1)$ in the case that $r + 1$ is prime, was obtained only very recently, as a (not quite direct) consequence of the "tight colored Tverberg theorem" below (announced in 2009, to be published 2014, see [7]). For this, a substantial change in the concepts of *colored* and *rainbow* was needed: We allow for more than $d+1$ colors, and a rainbow face does not need to use all the different colors, and instead of requiring that all color classes have size at least r, they now are required to have less than r elements (so, indeed, no color is used in all the blocks of a partition into r subsets). Here it is:

Tight colored Tverberg theorem [7]**:** *Let $d \geq 1$, r prime, $N \geq (r-1)(d+1)$, and $f : \Delta_N \longrightarrow \mathbb{R}^d$ continuous, where the $N+1$ vertices of Δ_N are colored and each color class C_i has size $|C_i| \leq r-1$. (Thus in particular there are at least $d+2$ different colors) Then Δ_N has r disjoint* rainbow faces *whose images under f intersect.*

Our original proof for this in [7] used equivariant obstruction theory. (It was rephrased in terms of degrees by Vrećica and Živaljević [14], with an error in the value given for the degree; it was also elaborated on by Matoušek, Tancer, and Wagner [11].) A substantially different proof via Fadell–Husseini index, with further applications, such as the finiteness of $N(r,d)$ for prime r, was provided in [8].

Colored versions via constraints. If one is content with a non-tight result, then indeed one can get a colored topological Tverberg theorem "nearly for free" from the original topological Tverberg theorem, by allowing for extra points but adding constraints, as follows. (See [6] for details.)

Lemma (A Tverberg unavoidable subcomplex)**.** *Let $d \geq 1$, let r be a prime power, and $N \geq (r-1)(d+1)$. If $f : \Delta_N \to \mathbb{R}^d$ is continuous, then for any set C of at most $2r-1$ vertices of Δ_N, every Tverberg r-partition for f has a block that has at most one vertex in C.*

Indeed, in this setting Tverberg r-partitions exist, and by the pigeonhole principle not all r blocks of a Tverberg r-partition can have at least two vertices in C.

Weak colored Tverberg theorem [6]**:** *Let $d \geq 1$, let r be a prime power, and $N \geq 2(r-1)(d+1)$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of Δ_N are colored by $d+1$ colors, where each color class C_i has cardinality at most $2r-1$, then there are r pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Proof. For $0 \leq i \leq d$, let $g_i : \Delta_N \to \mathbb{R}$ be the Euclidean distance from the subcomplex of Δ_N formed by the faces that have at most one vertex in the color class C_i. Now consider the continuous map

$$F := (f, g_0, \ldots, g_d) : \Delta_N \longrightarrow \mathbb{R}^{2d+1}.$$

According to the topological Tverberg theorem, this map F has a Tverberg r-partition into r vertex-disjoint faces $\sigma_1, \ldots, \sigma_r$, such that $f(x_1) = \cdots = f(x_r)$ for points $x_k \in \sigma_k$ in disjoint faces $\sigma_k \subset \Delta_N$, and such that $g_i(x_1) = \cdots = g_i(x_r)$ for $0 \leq i \leq d$. However, by the lemma for each color i one of the faces σ_k has only one vertex in C_i, that is, $g_i(x_k) = 0$, and this implies that *all* faces σ_k have only one vertex in C_i. □

Note that this "weak colored Tverberg theorem" is stronger than the one by Živaljević and Vrećica [16].

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