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Convergence of discrete holomorphic integrals on orthodiagonal discrete Riemann surfaces

Master Thesis
(Updated version)
by

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Contents

Introduction

This work is an updated version of my Master Thesis handed over on March 28, 2024. Note that there are still little unfilled gaps in this thesis. For example are the explanations in the introduction not completed. The importance of the Delaunay triangulation is not fully explained as well as the reasons why the attempt of Bobenko and Skopenkov is not directly transferred to orthodiagonal quadrilateral lattices. Furthermore, the conclusive chapter was entirely removed.

There is a long and varied history of linear theories of discrete complex analysis. Take a look at the work of Smirnov [Smi10], for example. The questions how to discretize the complex plane and how to define concepts of discrete holomorphic and harmonic functions on these discrete planes were often discussed with different answers. The simplest discrete complex planes are with certainty square lattices. The theory of discrete harmonic functions on these grids were studied in the 1920s by authors like Courant, Friedrichs and Lewy [RL28]. Here, and in [Duf53], a use of the early finite-element literature can be found. Isaac [Isa41] developed two different notations for discrete holomorphy on square lattices, and Duffin [Duf56; Duf68] extended the theory to rhombic lattices which is still a special case of quadrilateral lattices with orthogonally intersecting diagonals. Also triangular lattices were investigated, for example by Dynnikov and Novikov [DN03], and Stephenson [Ste05].
A Riemann surface is a two-real dimensional connected manifold with a complex structure on it. Mercat generalized the theory of discrete complex planes to Riemann surfaces via general quadrilaterals [Mer01b] and talked about discrete period matrices [Mer01a]. Later, Bobenko and Günther enhanced the theories for the planar case [BG16] and for Riemann surfaces [BG17] using general bipartite quadrilateral cellular decompositions. Using the medial graph their work build up discrete theorems to well known smooths theorems like the theorems of Liouville, Stokes and Riemann-Roch, and the Riemann-bilinear identity. In this thesis we consider bipartite quad-graphs as well.

Let $F : \Sigma \to \mathbb{R}$ be a function on a connected two-dimensional space Σ with fixed values at the boundary $\partial\Sigma$. We consider the Dirichlet energy functional

$$E_\Sigma(F) := \int_\Sigma |\nabla F| dxdy$$

for such functions. The minimizer of this functional is called harmonic. Inspired by [BG17] we formulate a discrete variant in Chapter 1.3. The boundary condition of the Dirichlet boundary problem for Riemann surfaces will be given by periods defined in Chapter 1.4.

Skopenkov [Sko13] investigated the Dirichlet boundary problem on finite simply connected quadrilateral lattices embedded in $\mathbb{C}$. He had shown that the solution is unique and that a sequence of discrete Dirichlet energies of discrete harmonic functions converges to the Dirichlet energy of the harmonic function on a domain Ω if the corresponding sequence of discrete surfaces converges to Ω. Furthermore, he had proven the convergence of these discrete harmonic functions to the smooth harmonic function if considering lattices of quadrilaterals with orthogonally intersecting diagonals, called orthodiagonal quadrilateral lattices.

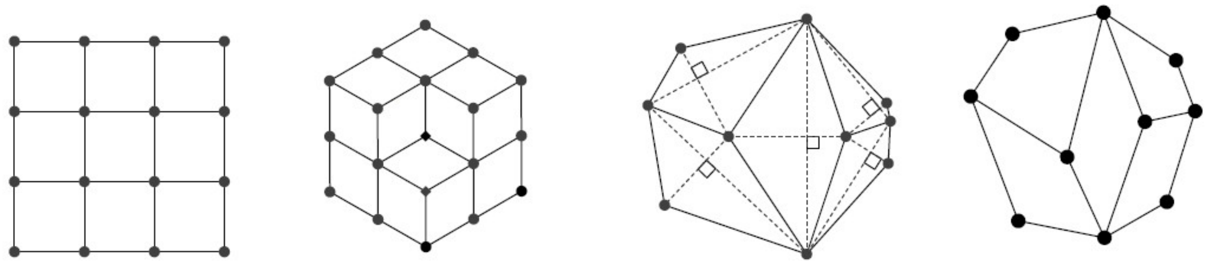

Figure 0.1: Examples of quadrilateral lattices (from the left to the right): square, rhombic, orthodiagonal, arbitrary (taken from [Sko13]).

Bobenko and Skopenkov continued to work on this topic in [BS16]. They considered discrete Riemann surfaces via triangular lattices and proved, with another concept of discrete harmonic and holomorphic functions, the convergence of the discrete Dirichlet energies and deduced the convergence of the discrete period matrices. They have also shown the convergence of discrete harmonic functions by using Delaunay triangulations as discrete surfaces. The special of this kind of triangulations is that we can build an orthodiagonal quadrilateral lattice, called Delaunay–Voronoi quadrangulation, where the triangle edges become diagonals. Together with the convergence of the period matrices they concluded the convergence of discrete holomorphic integrals.

Günther followed the theory of discrete Riemann surfaces via bipartite quad-graphs and proved the convergence of discrete Dirichlet energies and discrete period matrices in his work [Gün23]. To get the first convergence he took advice from nonconforming finite element methods [Bra07]. The main results are formulated and motivated in Chapter 1.5.
In this thesis we continue Günther's work and prove in Chapter 2 the convergence of discrete harmonic functions and in Chapter 3 the convergence of discrete holomorphic integrals. Similar as Bobenko and Skopenkov in [BS16] with the Delauney triangulations and Skopenkov in [Sko13], we need to consider orthodiagonal discrete Riemann surfaces, especially a nondegenerate uniform sequence. In Chapter 1 we clear the basic definitions and statements we need where we orientate us at [BG17] and [Gün23] focusing on compact polyhedral surfaces. The proofs of the convergence theorems Theorem 2.1 and Theorem 3.1 are inspired by [Sko13] and [BS16].
The orthodiagonality leads to some useful properties for discrete harmonic functions. Each function f on a bipartite quad-graph can be split into a function f^B on the black vertices and f^W on the white vertices. By Proposition 1.5 we get also a split

$$\Delta f = \Delta f^B + \Delta f^W$$

for the discrete Laplacian. Therefore, in Section 1.3.1 we get an orthodiagonal version of the discrete Green's identity. Another important property for discrete harmonic functions on orthodiagonal lattices is the Maximum principle (Lemma 2.2) which says that a discrete harmonic function on an orthodiagonal quad-surface with boundary becomes maximal (and also minimal) at the boundary. These statements play an important role in our proof of Theorem 2.1. Similar as in [BS16], the proof of Theorem 3.1 will be a short consequence from the convergence of discrete harmonic functions in combination with the convergence of the discrete period matrices for orthodiagonal discrete Riemann surfaces proven in [Gün23] (see Corollary 1.14).

Chapter 1

Discrete Riemann surfaces

In this chapter we characterize Riemann surfaces as polyhedral surfaces and discretize them by using bipartite quad-graphs as in [BG16] and [BG17]. In Section 1.3 we will define discrete holomorphic and harmonic functions and differentials, the discrete Dirichlet energy operator and the discrete Laplacian operator on discrete Riemann surfaces. Later, in Section 1.4, we classify periods of discrete one-forms, and of discrete multi-valued functions on the universal covering of the Riemann surface. At the end of this chapter, in Section 1.5, we formulate the convergences of discrete Dirichlet energies and discrete period matrices to their smooth counterparts, as proven in [Gün23].

1.1 Polyhedral surfaces and conical singularities

Talking about Riemann surfaces, as in [Gün23] or [BS16], in this thesis we consider *polyhedral surfaces* Σ without boundary those are not homeomorphic to the sphere, or simply connected subsets of a polyhedral surface (e.g. a simply connected union of quadrilaterals embedded in $\mathbb{C}$, called a *quad-surfaced polygon*). This means that Σ is equipped with a piecewise flat metric having isolated conical singularities, i.e., vertices where the interior angles of incident polygons do not sum up to 2π.

We denote the *intrinsic distance* of two point $z, w \in \Sigma$ by $|zw|$, or by $|z-w|$ if the direct line connecting z and w lies fully in Σ, the *genus* of Σ by $g \in \mathbb{N}$. More about the genus of a compact Riemann surface can be read in [Bob11].

Definition. Let S denote the (finite) *set of conical singularities* of Σ. For $O \in S$ the *singularity index* is the positive real number γ_O such that $2\pi/\gamma_O$ equals the sum of angles at O of all polygons incident to this vertex. For $O \in \Sigma \setminus S$ we define $\gamma_O := 1$.
For $O \in \Sigma$ we denote by D_O an intrinsic open disk of sufficiently small radius R_O around O such that its closure does not contain any further singularity than O. We call $D_O \subset \Sigma$ an *intrinsic cone* with apex O.

Remark. Any compact Riemann surface can be realized as a polyhedral surface (see Troyanov [Tro86]).

1.2 Definition of discrete Riemann surfaces

After we have clarified the concept of Riemann surfaces, we will define discrete Riemann surfaces via bipartite quad-graphs. We are following the notation from [Gün23].

Definition. A *discrete Riemann surface* (Σ, Λ) - or for short Λ - is a strongly regular finite decomposition - called *quad-graph* - of Σ into flat embedded quadrilaterals $F(\Lambda)$ - called *faces* of Λ - where $V(\Lambda)$ and $E(\Lambda)$ are the sets of *vertices* and *edges* of Λ such that $S \subset V(\Lambda)$ and $(V(\Lambda), E(\Lambda))$ is bipartite.

The dual graph of Λ is denoted by $\Diamond$. The diagonals of $F(\Lambda)$ give rise to two connected graphs Γ, Γ^* that are dual to each other. We call Γ the *black graph* and its dual Γ^* the *white graph* of Λ. If the diagonals of each quadrilateral intersect orthogonally, we call (Σ, Λ) an *orthodiagonal discrete Riemann surface.* A quad-surfaced polygon is considered as a *discrete Riemann surface with boundary* $\partial\Lambda$.

Definition. Let $Q \in F(\Lambda)$. We denote by b_-, w_-, b_+, w_+ its vertices in counterclockwise order, starting with a black vertex, and identify them with their complex values given by an isometry z_Q of Q into $\mathbb{C}$. Define

$$\rho_Q := -i\frac{w_+ - w_-}{b_+ - b_-}.$$

The intersection angle of the diagonals is given by

$$\varphi_Q := \arccos\left(\operatorname{Re}\left(i\frac{\rho_Q}{|\rho_Q|}\right)\right).$$

We call z_Q *discrete chart* (of Q). The set of all ρ_Q, $Q \in F(\Lambda)$, defines the *discrete complex structure* of (Σ, Λ).

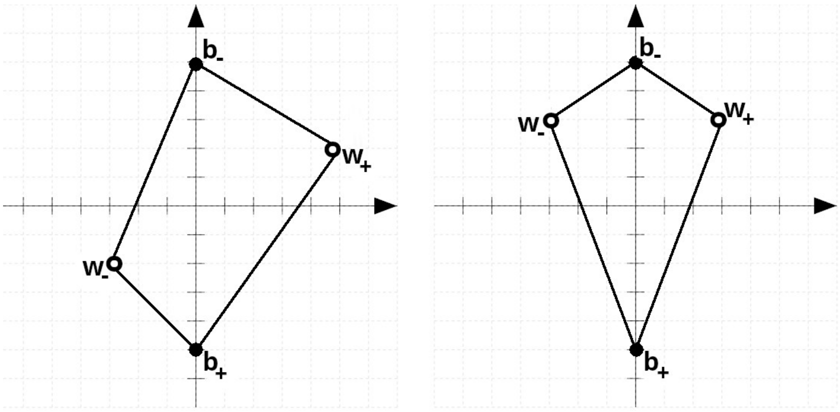

Figure 1.1: First (left) and second (right) part of Proposition 1.1.

Proposition 1.1. It is $\operatorname{Re}(\rho_Q) \in \mathbb{R}^+$ for all $Q \in F(\Lambda)$. If the diagonals of Q intersect orthogonally, it holds $\rho_Q \in \mathbb{R}^+$.

Proof. Let b_-, w_-, b_+, w_+ be the vertices of Q as above. W.l.o.g. let $b_- = \frac{a}{2}i$, $b_+ = -\frac{a}{2}i$, $a > 0$, i.e., $\operatorname{Re}(w_-) < \operatorname{Re}(w_+)$ (the quadrilateral does not need to be convex). Therefore, we have $\rho_Q = \frac{w_+ - w_-}{a}$ and $\operatorname{Re}(w_+ - w_-) > 0$, thus we get $\operatorname{Re}(\rho_Q) = \operatorname{Re}(\frac{w_+ - w_-}{a}) > 0$. If the diagonals intersect orthogonally, we get further $\operatorname{Im}(w_-) = \operatorname{Im}(w_+)$ and consequently $\operatorname{Im}(\rho_Q) = \operatorname{Im}(\frac{w_+ - w_-}{a}) = 0$. $\square$

Remark. It was proven in [BG17], Theorem 2.2, that not all discrete Riemann surfaces correspond to polyhedral surfaces. It is sufficient that each ρ_Q, $Q \in F(\Lambda)$, is real, meaning each orthodiagonal discrete Riemann surface corresponds to a polyhedral surface.

Definition. Let $v \in V(\Lambda)$. The *star* of v is the finite collection of all $Q \in F(\Lambda)$ such that v is a vertex of Q. We denote the star of v by $\operatorname{star}(v)$. Furthermore, we denote for each $Q_s \in \operatorname{star}(v)$ the vertices in counterclockwise order by v, v'_{s-1}, v_s, v'_s. We denote by z_v a *discrete chart* of $\operatorname{star}(v)$.

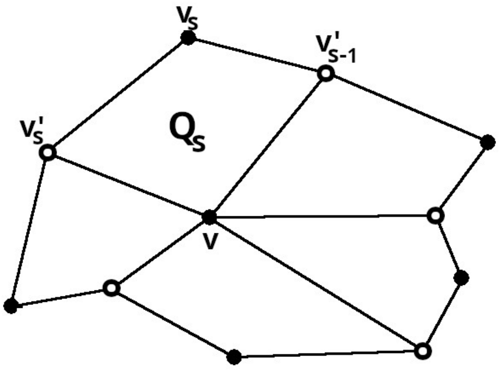

Figure 1.2: The star of a non-boundary vertex $v \in V(\Lambda)$.

Definition. The *medial graph* X of Λ is defined to be the following graph.

- Its vertex set $V(X)$ consists of the midpoints of any $e \in E(\Lambda)$.
- Its edge set $E(X)$ consists of all lines $[Q, v]$ connecting midpoints of edges of Λ that share a vertex $v \in V(\Lambda)$ and belong to the same quadrilateral $Q \in F(\Lambda)$. We call an edge $[Q, v]$ *black* if it is parallel to the black diagonal of Q, otherwise we call this edge *white*.
- A face $F \in F(X)$ is bounded by all edges $[Q, v]$ where we either consider all $Q \in \mathrm{star}(v)$, $v \notin V(\partial\Lambda)$, or all $v \in V(Q)$. We denote these faces either by F_v or F_Q.

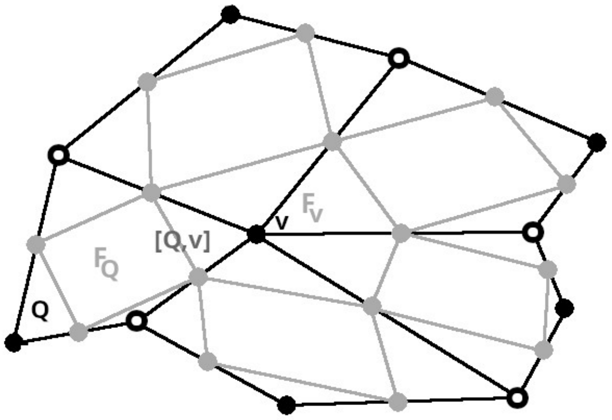

Figure 1.3: The medial graph of $\mathrm{star}(v)$.

The following definition is used in [Gün23]. For reasons of notation, we will also provide an equivalent definition.

Definition. Let $\Phi \in]0, \frac{\pi}{2}]$. We call a discrete Riemann surface (Σ, Λ) Φ-*regular* if the interior angles of each quadrilateral $Q \in F(\Lambda)$ are bounded from below by Φ and if the intersection angle between the diagonals is always in the interval $[\Phi, \pi - \Phi]$.

Remark. The property of Φ-regularity for a given $\Phi \in]0, \frac{\pi}{2}]$ implies the existence of a constant $\mathrm{Const}_\Phi \in \mathbb{R}$ depending on Φ such that

$$\frac{1}{\mathrm{Const}_\Phi} < \frac{|b_+ - b_-|}{|w_+ - w_-|} < \mathrm{Const}_\Phi \tag{1.1}$$

holds for any quadrilateral $Q \in F(\Lambda)$ with black vertices b_+, b_- and white vertices w_+, w_-.

Definition. A sequence of discrete Riemann surfaces $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ is called *nondegenerate uniform* if the following two conditions are satisfied.

- For a given $\Phi \in]0, \frac{\pi}{2}]$ the discrete Riemann surface (Σ, Λ_n) is Φ-regular for any $n \in \mathbb{N}$.
- Let $p \in \Sigma$. Then, there is a constant $C \in \mathbb{N}$ independent of n such that the number of vertices of Λ_n in an arbitrary intrinsic disk around p of radius equal to the maximal edge length of (Σ, Λ_n) is smaller than C.

We say that this sequence converges to the smooth Riemann surface Σ as $n \to \infty$ if the maximal edge length of Λ_n tends to zero.

1.3 Functions on discrete Riemann surfaces

In this section we define holomorphic functions, discrete one- and two-forms, the discrete Hodge star and the discrete Laplacian operator. Furthermore, we will define the discrete Dirichlet energy functional and state the discrete Stokes' theorem.
We close this section with a general and an orthodiagonal version of the discrete Green's identity.

Let (Σ, Λ) be a discrete Riemann surface.

Definition. A function $f : V(\Lambda) \to \mathbb{C}$ is called *discrete holomorphic* if

$$f(w_+) - f(w_-) = i\rho_Q(f(b_+) - f(b_-)) \tag{1.2}$$

is satisfied for all quadrilaterals $Q \in F(\Lambda)$ with vertices b_-, w_-, b_+, w_+ in counterclockwise order, starting with a black vertex. We call f *discrete antiholomorphic* if $\bar{f}$ is discrete holomorphic.

Definition. Let $f : V(\Lambda) \to \mathbb{R}$ and $Q \in F(\Lambda)$ with vertices b_-, w_-, b_+, w_+ in counterclockwise order identified with their complex values through a chart z_Q. The *discrete gradient* of f at Q is defined as the unique vector $\nabla_Q f \in \mathbb{R}^2$ such that

$$\begin{aligned} \nabla_Q f \cdot \overrightarrow{b_- b_+} &= f(b_+) - f(b_-), \\ \nabla_Q f \cdot \overrightarrow{w_- w_+} &= f(w_+) - f(w_-), \end{aligned}$$

where $\overrightarrow{xy}$ for x, y in $\mathbb{C}$ or $\mathbb{R}^2$ describes the vector from x to y in $\mathbb{R}^2$, and $\cdot$ describes the standard Euclidean scalar product of $\mathbb{R}^2$.
The *discrete Dirichlet energy* of f is given by

$$E_\Lambda(f) := \int_{F(\Lambda)} \nabla_Q f \cdot \nabla_Q f = \sum_{Q\in F(\Lambda)} |\nabla_Q f|^2 \mathrm{area}(Q). \tag{1.3}$$

The definition in (1.3) follows the definition of the *classical Dirichlet energy*

$$E_\Sigma(F) := \int_\Sigma |\nabla F|^2 dxdy$$

for continuous piecewise-smooth functions $F : \Sigma \to \mathbb{R}$ on Σ.

Remark. In the context of quadrilaterals Q with vertices b_-, w_-, b_+, w_+ it holds

$$\text{area}(Q) = \frac{1}{2}|b_+ - b_-||w_+ - w_-| \cdot \sin(\varphi_Q), \tag{1.4}$$

where φ_Q is the angle of the diagonals of Q

Definition. A *discrete one-form* ω is a complex function on the oriented edges of the medial graph X such that $\omega(-e) = -\omega(e)$ holds for any oriented edge e of X. Here, $-e$ denotes the edge e with opposite direction. We denote

$$\int_e \omega := \omega(e).$$

For a finite collection $P = \{e_1, ..., e_n\}$ of oriented edges of X we define the *discrete integral* of ω over P as

$$\int_P \omega := \sum_{k=1}^n \int_{e_k} \omega.$$

We write $\oint_P \omega$ if P is a closed path in X.
We say that ω is of *type* $\Diamond$ if for each $Q \in F(\Lambda)$ and its incident same coloured vertices v_-, v_+ the equality $\omega([Q, v_-]) = -\omega([Q, v_+])$ holds. The orientation of these two edges is given by the orientation of ∂F_Q.
Let z be a chart of a quadrilateral $Q \in F(\Lambda)$. On the oriented edges e of $F_Q \in F(X)$, the discrete one-forms dz and $d\bar{z}$ are defined by $\int_e dz = z(e)$ and $\int_e d\bar{z} = \overline{z(e)}$, where $z(e)$ is the difference of end and starting point of e.

Remark. A discrete one-form of type $\Diamond$ can be locally represented as $pdz_Q + qd\bar{z}_Q$ on F_Q for $p, q \in \mathbb{C}$ depending only on Q.

Definition. A *discrete two-form* Ω is a complex function on $F(X)$. For all $F \in F(X)$ we denote

$$\iint_F \Omega := \Omega(F).$$

For a finite set $Z = \{F_1, ..., F_n\} \subseteq F(X)$ we define the *discrete integral* of Ω over Z as

$$\iint_Z \Omega := \sum_{k=1}^n \iint_{F_k} \Omega.$$

We say that Ω is of *type* Λ if Ω vanishes on all faces corresponding to faces of Λ, and that Ω is of *type* $\Diamond$ if Ω vanishes on all faces corresponding to vertices of Λ.
Let z be a chart of either a quadrilateral $Q \in F(\Lambda)$ or a non-boundary vertex $v \in V(\Lambda)$. The discrete two-form $dz \wedge d\bar{z}$ is defined by $\iint_F dz \wedge d\bar{z} = -4i\text{area}(z(F))$, where $F = F_Q$ or $F = F_v$, respectively.
We split $dz \wedge d\bar{z} =: \Omega_\Lambda^z + \Omega_\Diamond^z$, where Ω_Λ^z becomes zero if z is a map of a face $Q \in F(\Lambda)$, and otherwise $\Omega_\Diamond^z$ becomes zero.

Remark. For $Q \in F(\Lambda)$ it holds $2\text{area}(z_Q(F_Q)) = \text{area}(Q)$.

Definition. Let $f : V(\Lambda) \to \mathbb{C}$, $h : F(\Lambda) \to \mathbb{C}$, let ω be any discrete one-form and let Ω_1, Ω_2 be discrete two-forms of type Λ and $\Diamond$, respectively. For any oriented edge $e = [Q, v]$ and any faces $F_v, F_Q \in F(X)$ corresponding to $v \in V(\Lambda)$ and $Q \in F(\Lambda)$, we define the products $f\omega$, $h\omega$, $f\Omega_1$ and $h\Omega_2$ by

$$\begin{aligned} \int_e f\omega &:= f(v)\int_e \omega; & \iint_{F_v} f\Omega_1 &:= f(v)\iint_{F_v}\Omega_1, \ \iint_{F_Q} f\Omega_1 := 0; \\ \int_e h\omega &:= h(Q)\int_e \omega; & \iint_{F_v} h\Omega_2 &:= 0, \ \iint_{F_Q} h\Omega_2 := h(Q)\iint_{F_Q}\Omega_2. \end{aligned}$$

Definition. Let $Q \in F(\Lambda)$ and $v \in V(\Lambda)$ be a non-boundary vertex. Furthermore, let f be a complex function on the vertices of Q and let h be a complex function on the quadrilaterals of $\text{star}(v)$. We consider the faces F_Q and F_v with counterclockwise orientation of their boundaries. The *discrete derivatives* $\partial_\Lambda f$, $\bar{\partial}_\Lambda f$ in the chart z_Q and $\partial_\Diamond h$, $\bar{\partial}_\Diamond h$ in the chart z_v are defined by

$$\begin{aligned} \partial_\Lambda f(Q) &:= \frac{-1}{2i\text{area}(Q)}\oint_{\partial F_Q} f d\bar{z}_Q, & \bar{\partial}_\Lambda f(Q) &:= \frac{1}{2i\text{area}(Q)}\oint_{\partial F_Q} f dz_Q; \\ \partial_\Diamond h(v) &:= \frac{-1}{4i\text{area}(z_v(F_v))}\oint_{\partial F_v} h d\bar{z}_v, & \bar{\partial}_\Diamond h(v) &:= \frac{1}{4i\text{area}(z_v(F_v))}\oint_{\partial F_v} h dz_v. \end{aligned}$$

We call h *discrete holomorphic* in the chart z_v if $\bar{\partial}_\Diamond h(v) = 0$.

Remark. It is f discrete holomorphic in z_Q if and only if $\bar{\partial}_\Lambda f(Q)$ equals zero. This is a consequence of Theorem 1.2 and equality (1.2).

Definition. Let $f : V(\Lambda) \to \mathbb{C}$ and $h : F(\Lambda) \to \mathbb{C}$. The *discrete exterior derivatives* df and dh on the edges of X in a chart z are defined by

$$df := \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z}, \quad dh := \partial_\Diamond h dz + \bar{\partial}_\Diamond h d\bar{z}.$$

Let ω be a discrete one-form defined on all edges of $F_v \in F(X)$ corresponding to $v \in V(\Lambda)$ or of $F_Q \in F(X)$ corresponding to $Q \in F(\Lambda)$. Further, let z be a discrete chart of v or Q, respectively. We write $\omega = pdz + qd\bar{z}$ with functions p, q defined on the faces incident to v or vertices incident to Q, respectively. The *discrete exterior derivative* $d\omega$ is given by

$$\begin{aligned} d\omega|_{F_v} &:= (\partial_\Diamond q - \bar{\partial}_\Diamond p)\Omega_\Lambda^z, \\ d\omega|_{F_Q} &:= (\partial_\Lambda q - \bar{\partial}_\Lambda p)\Omega_\Diamond^z. \end{aligned}$$

Definition. A discrete one-form ω is called *closed* if $d\omega = 0$.

The *discrete Stokes' theorem* concludes the independence from the chosen chart and from the choice of p and q and motivates the definition above. A proof can be found in [BG16], Theorem 2.3.1.

Theorem 1.2 (Discrete Stokes')**.** Let $f : V(\Lambda) \to \mathbb{C}$ and let ω be a discrete one-form. Then, for any directed edge e of X starting in the midpoint of the edge vv'_- and ending in the midpoint of the edge vv'_+ of a quadrilateral $Q \in F(\Lambda)$, and for any finite collection of faces Z of X with counterclockwise oriented boundary ∂Z we have

$$\int_e df = \frac{f(v'_+) - f(v'_-)}{2} = \frac{f(v) + f(v'_+)}{2} - \frac{f(v) + f(v'_-)}{2},$$
$$\iint_Z d\omega = \oint_{\partial Z} \omega.$$

Remark. For $f : V(\Lambda) \to \mathbb{C}$, df is - as consequence of Theorem 1.2 - a closed one-form of type $\Diamond$.

Remark. By use of the discrete Stokes' theorem we get an alternative definition for the discrete gradient of a function $f : V(\Lambda) \to \mathbb{R}$. We can define $\nabla_Q f \in \mathbb{R}^2$ as the unique vector satisfying

$$\nabla_Q f \cdot z_Q(e) = \int_e df$$

for all edges e of F_Q. Note that an edge e of a quadrilateral F_Q is half as long as the corresponding parallel diagonal of Q.

Definition. Let ω, ω' be two discrete one-forms of type $\Diamond$, i.e., in a chart z_Q of $Q \in F(\Lambda)$ there are unique $p, q, p', q' \in \mathbb{C}$ depending on Q such that

$$\omega|_{\partial F_Q} = p dz_Q + q d\bar{z}_Q, \quad \omega'|_{\partial F_Q} = p' dz_Q + q' d\bar{z}_Q.$$

The *discrete wedge product* of ω and ω' is the discrete two-form of type $\Diamond$ defined through

$$\omega \wedge \omega'|_{F_Q} := (pq' - qp')\Omega_{\Diamond}^{z_Q}.$$

Definition. Let $f : V(\Lambda) \to \mathbb{C}$, $h : F(\Lambda) \to \mathbb{C}$, let ω be a discrete one-form of type $\Diamond$ and let Ω be a discrete two-form either of type Λ or $\Diamond$. For each $Q \in F(\Lambda)$ there are $p, q \in \mathbb{C}$ such that $\omega|_{\partial F_Q} = p dz_Q + q d\bar{z}_Q$. The *discrete Hodge star* of ω is defined by

$$\star\omega|_{\partial F_Q} := -ip dz_Q + iq d\bar{z}_Q.$$

Let Ω_Σ be the discrete two-form such that $\Omega_\Sigma(F) = 1$ holds for all $F \in F(X)$. The *discrete Hodge star* of f, h and Ω is defined through the discrete two-forms $\star f := f\Omega_\Sigma$ of type Λ, the discrete two-form $\star h := h\Omega_\Sigma$ of type $\Diamond$ and the discrete function $\star\Omega := \frac{\Omega}{\Omega_\Sigma}$ defined on $V(\Lambda)$ or $F(\Lambda)$, respectively.
It follows that $\star f(F_v) = f(v)$ and $\star h(F_Q) = h(Q)$ holds for any non-boundary $v \in V(\Lambda)$ and $Q \in F(\Lambda)$, and vice versa.

Remark. By definition, for complex functions on $V(\Lambda)$ or $F(\Lambda)$ and for two-forms either of type Λ or $\Diamond$ we have $\star^2 = \mathrm{Id}$. For a discrete one-form of type $\Diamond$ we have $\star^2 = -\mathrm{Id}$.

Definition. The *discrete scalar product* of two discrete one-forms ω, ω' of type $\Diamond$ is defined by

$$\langle \omega, \omega' \rangle := \iint_{F(X)} \omega \wedge \star\overline{\omega'}.$$

Note that we just integrate over the faces F_Q corresponding to faces $Q \in F(\Lambda)$ by definition of the discrete wedge product.
In a similar way, we define the discrete scalar product of $f, g : V(\Lambda) \to \mathbb{C}$ by

$$\langle f, g \rangle := \iint_{F(X)} f \star \bar{g},$$

where $f \star \bar{g}$ is defined to be zero at faces of X corresponding to faces of of Λ, and of two discrete two-forms Ω, Ω' on $F(X)$ by

$$\langle \Omega, \Omega' \rangle := \iint_{F(X)} \Omega \star \overline{\Omega'}.$$

The *formal adjoint* of the exterior derivation operator d regarding these scalar products is given by $\delta := - \star d \star$, as seen in [BG17], Proposition 4.10.

Definition. The *discrete Laplacian operator* for functions $f : V(\Lambda) \to \mathbb{C}$, discrete one-forms ω of type $\Diamond$ or discrete two-forms of type Λ is defined as the linear operator

$$\Delta := -\delta d - d\delta = \star d \star d + d \star d \star,$$

where $d\Omega$ for a discrete two-form Ω is defined to be zero.
We call f *discrete harmonic* at $v \in V(\Lambda)$ if $\Delta f(v) = 0$ holds, and we call ω *discrete harmonic* if $\Delta\omega = 0$ holds, i.e., if and only if $d\omega = 0$ and $d \star \omega = 0$ hold.
We denote the *classical Laplacian Operator* on Σ by Δ_Σ or, alternatively, by $\Delta_\mathbb{C}$ if Σ is a subset of $\mathbb{C}$ like a quad-surfaced polygon, for example.

Remark. We have also seen in [BG17] that $\Delta f(v)$ is proportional to $4\partial_\Diamond \bar{\partial}_\Lambda f(v) = 4\bar{\partial}_\Diamond \partial_\Lambda f(v)$ in a chart z_v around $v \in V(\Lambda)$. In particular, discrete holomorphy of $f : V(\Lambda) \to \mathbb{C}$ implies discrete harmonicity. Furthermore, $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ inherit discrete harmonicity because of $\mathrm{Re}(\Delta f) = \Delta \mathrm{Re}(f)$ and $\mathrm{Im}(\Delta f) = \Delta \mathrm{Im}(f)$.

In the following we get an equivalent definition for the discrete Dirichlet energy. In comparison to the proof in [Gün23], Lemma 2.7, we show more calculations.

Proposition 1.3. Let $f : V(\Lambda) \to \mathbb{R}$. It holds $E_\Lambda(f) = \langle df, df \rangle$.

Proof. It is only to show that

$$\iint_{F_Q} df \wedge \star d\bar{f} = |\nabla_Q f|^2 \mathrm{area}(Q)$$

holds for any $Q \in F(\Lambda)$.
So let $Q \in F(\Lambda)$. By using $df := \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z}$ and $f = \bar{f}$ we get $\star d\bar{f} = -i\partial_\Lambda f dz + i\bar{\partial}_\Lambda f d\bar{z}$, and therefore

$$df \wedge \star d\bar{f} = (i\partial_\Lambda f \bar{\partial}_\Lambda f + \bar{\partial}_\Lambda f i \partial_\Lambda f)\Omega^z_\Diamond$$

in a chart $z = z_Q$ of Q. Hence, we calculate

$$\iint_{F_Q} df \wedge \star d\bar{f} = 4|\partial_\Lambda f|^2 \mathrm{area}(Q).$$

At next let us integrate $\nabla_Q f$ on Q to a linear function F on Q. Denote by g the restriction of F to the vertices of Q. Therefore, we get $df = dg$. We have seen in [BG16] that $\partial_\Lambda g$ agrees with the smooth derivative of F. In particular,

$$4|\partial_\Lambda f|^2 = 4|\partial_\Lambda g|^2 = 4|\partial F|^2 = |\nabla F|^2 = |\nabla_Q f|^2.$$

This ends the proof. □

The following proposition will be helpful for eventual calculations. A proof can be found in [BG16], Proposition 2.3.10.

Proposition 1.4. Let ω be a discrete one-form of type $\Diamond$, let $Q \in F(\Lambda)$, and let e, e^* be oriented edges of X parallel to the black and white diagonal of Q, respectively, such that $\operatorname{Im}(\frac{e^*}{e}) > 0$. Then

$$\int_e \star\omega = \cot(\varphi_Q) \int_e \omega - \frac{|e|}{|e^*| \sin(\varphi_Q)} \int_{e^*} \omega,$$
$$\int_{e^*} \star\omega = \frac{|e^*|}{|e| \sin(\varphi_Q)} \int_e \omega - \cot(\varphi_Q) \int_{e^*} \omega.$$

Remark. In the case that the diagonals of $Q \in F(\Lambda)$ intersect orthogonally, we get $\cot(\varphi_Q) = 0$, thus the Hodge star operator makes an integral over a black edge of X to an integral over a white edge of X and vice versa.

To prove the Maximum Principle in Chapter 2.1, we need the following property of the discrete Laplacian operator. A proof can be found in [BG16], Corollary 2.4.1.

Proposition 1.5. Let $f : V(\Lambda) \to \mathbb{C}$ and $v \in V(\Lambda)$. Then

$$\Delta f(v) = \frac{1}{4\text{area}(F_v)} \sum_{Q_s \in \text{star}(v)} \frac{1}{\operatorname{Re}(\rho_s)} \left(|\rho_s|^2 (f(v_s) - f(v)) + \operatorname{Im}(\rho_s)(f(v'_s) - f(v'_{s-1}))\right),$$

where $\rho_s := \rho_{Q_s}$ for $v \in V(\Gamma)$ or $\rho_s := \frac{1}{\rho_{Q_s}}$ otherwise.

The following statement can be found in [BG17], Lemma 4.11. It shows us that discrete harmonic functions are critical points of the *discrete Dirichlet energy functional.* A proof of the planar case can be found in [BG16], Lemma 2.4.9.

Proposition 1.6. The discrete Dirichlet energy functional $E_\Lambda(f)$ for functions $f : V(\Lambda) \to \mathbb{R}$ is a convex nonnegative quadratic functional in the vector space of real functions on $V(\Lambda)$. Furthermore,

$$-\frac{\partial E_\Lambda}{\partial f(v)}(f) = 2\Delta f(v) \iint_{F_v} \Omega_\Sigma$$

holds for any non-boundary $v \in V(\Lambda)$. We call this functional's critical points *discrete harmonic.*
If we fix the input functions at $V(\partial\Lambda)$, the so called *Dirichlet boundary value problem* is uniquely solvable.

1.3.1 Discrete Green's identity

In the following we will state a general and an orthodiagonal version of the *discrete Green's identity.* The orthodiagonal version will be important for the proof of the convergence of discrete harmonic functions to their smooth counterpart under the convergence of a nondegenerate uniform sequence of discrete Riemann surfaces in Chapter 2.

First, we will set up a discrete product rule for exterior derivatives, proven in [BG16], Theorem 2.3.8.

Proposition 1.7. Let $f : V(\Lambda) \to \mathbb{C}$ and let ω be a discrete one-form of type $\Diamond$. Then, we get $d(f\omega) = df \wedge \omega + f d\omega$.

Now, let (Σ, Λ) be a quad-surfaced polygon, i.e., $\partial X \neq \emptyset$. The following proposition can be found in [BG16], Theorem 2.4.4.

Proposition 1.8 (Green's identity)**.** Let $f, g : V(\Lambda) \to \mathbb{C}$. It holds

(i) $\langle f, \Delta g \rangle + \langle df, dg \rangle = \oint_{\partial X} f \star d\bar{g}$,

(ii) $\langle f, \Delta g\rangle - \langle \Delta f, g\rangle = \oint_{\partial X}(f \star d\bar{g} - \bar{g} \star df)$.

Proof. (i) By using Proposition 1.7 and $\star^2 = \mathrm{Id}$ for discrete two-forms, we get

$$d(f \star d\bar{g}) = df \wedge \star d\bar{g} + f \star (\star d \star d\bar{g}) = df \wedge \star d\bar{g} + f \star (\Delta \bar{g}).$$

Using the discrete Stokes' Theorem on the left term yields the desired result.

(ii) This is a straight consequence of (i) by using the symmetry of the discrete scalar product. □

Corollary 1.9 (Orthodiagonal Green's identity)**.** Let (Σ, Λ) be orthodiagonal and let $f, g : V(\Lambda) \to \mathbb{C}$. Then, we get

$$\sum_{z \in V(\Gamma)} (f\Delta\bar{g} - \bar{g}\Delta f)(z) = \oint_{\partial X} (f^B \star d\bar{g}^B - \bar{g}^B \star df^B). \tag{1.5}$$

Before we start with the proof, we should make a few remarks. If we consider $h, k : V(\Lambda) \to \mathbb{C}$, we get

$$\langle h, k\rangle = \sum_{v \in V(\Lambda)} \iint_{F_v} h \star \bar{k} = \sum_{v \in V(\Lambda)} h(v) \iint_{F_v} \star \bar{k} = \sum_{v \in V(\Lambda)} h(v)\bar{k}(v).$$

By restricting the scalar product to faces of X corresponding to vertices of Γ, we get the left-hand side of (1.5) from the left-hand side of Proposition 1.8 (ii).
The idea of the proof is to split $f = f^B + f^W$ and $g = g^B + g^W$, where $f^B, g^B : V(\Lambda) \to \mathbb{C}$ vanish on the white vertices and $f^W, g^W : V(\Lambda) \to \mathbb{C}$ vanish on the black vertices of Λ. By Proposition 1.5 and the orthodiagonality of Λ we get $\Delta f = \Delta f^B + \Delta f^W$ and $\Delta g = \Delta g^B + \Delta g^W$.

Proof. If we split f and g into its black and white components, the left-hand side of Proposition 1.8 (ii) becomes

$$\langle f^B, \Delta g^B\rangle - \langle \Delta f^B, g^B\rangle + \langle f^W, \Delta g^W\rangle - \langle \Delta f^W, g^W\rangle = \sum_{z \in V(\Gamma)} (f\Delta\bar{g} - \bar{g}\Delta f)(z) + \sum_{z \in V(\Gamma^*)} (f\Delta\bar{g} - \bar{g}\Delta f)(z).$$

Now, we examine the right-hand side. We get

$$f \star d\bar{g} = f^B \star d\bar{g}^B + f^B \star d\bar{g}^W + f^W \star d\bar{g}^B + f^W \star d\bar{g}^W.$$

The mixed terms vanish in the integral since

$$\begin{aligned}
\oint_{\partial X} f^B \star d\bar{g}^W &= \sum_{\substack{[Q,v] \in \partial X,\\ v \in V(\Gamma)}} \int_{[Q,v]} f^B \star d\bar{g}^W + \sum_{\substack{[Q,v] \in \partial X,\\ v \in V(\Gamma^*)}} \int_{[Q,v]} f^B \star d\bar{g}^W \\
&= \sum_{\substack{[Q,v] \in \partial X,\\ v \in V(\Gamma)}} f^B(v) \int_{[Q,v]} \star d\bar{g}^W + \sum_{\substack{[Q,v] \in \partial X,\\ v \in V(\Gamma^*)}} f^B(v) \int_{[Q,v]} \star d\bar{g}^W \\
&= \sum_{\substack{[Q,v] \in \partial X,\\ v \in V(\Gamma)}} f^B(v) \int_{[Q,v]} \star d\bar{g}^W.
\end{aligned}$$

By Proposition 1.4 the integral $\int_{[Q,v]} \star d\bar{g}^W$ is a linear combination of $\cot(\varphi_Q) \int_{[Q,v]} d\bar{g}^W$ and $\int_{[Q,v]^*} d\bar{g}^W$, where $[Q, v]^*$ is a corresponding edge of X parallel to the black diagonal of Q. By the orthodiagonality we get $\cot(\varphi_Q) = 0$. Because of the discrete Stokes' Theorem and the fact that g^W vanishes on black vertices, $\int_{[Q,v]^*} d\bar{g}^W$ vanishes. All in all, we get

$$\oint_{\partial X} f^B \star d\bar{g}^W = 0 = \oint_{\partial X} f^W \star d\bar{g}^B.$$

Hence, the right-hand side of Proposition 1.8 (ii) becomes

$$\oint_{\partial X} (f^B \star d\bar{g}^B - \bar{g}^B \star df^B) + \oint_{\partial X} (f^W \star d\bar{g}^W - \bar{g}^W \star df^W),$$

where the first integral is zero on edges $[Q, v]$ of X regarding vertices $v \in V(\Gamma^*)$ and the second integral is zero regarding vertices $v \in V(\Gamma)$.
This concludes the proof. □

1.4 Periods and multi-valued functions

Let Σ be a polyhedral surfaces of genus $g \in \mathbb{N} \setminus \{0\}$ and (Σ, Λ) a discrete Riemann surface. In this section we define periods of discrete one-forms and extend them to so called multi-valued functions.

Let $p : \widetilde{\Sigma} \to \Sigma$ denote the *universal covering* of Σ. This leads us to a non-compact discrete Riemann surface $(\widetilde{\Sigma}, \widetilde{\Lambda})$ with medial graph $\widetilde{X}$.
Let $\widetilde{v}_0 \in V(\widetilde{\Lambda})$ be fixed and let $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$ be continuous loops on Σ with base point $v_0 := p(\widetilde{v}_0)$ such that these loops cut out a fundamental $4g$-gon F_g. The order of the loops at ∂F_g is

$$\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, ..., \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1}.$$

Their *homology classes* $a_1, ..., a_g, b_1, ..., b_g$ form a *canonical homology basis* of the *first homology group* $H_1(\Sigma, \mathbb{Z})$ of Σ, as seen in [Bob11]. Denote by $\alpha'_1, ..., \alpha'_g, \beta'_1, ..., \beta'_g$ closed paths on X with fixed base point $x_0 \in V(X)$ and having homologies $a_1, ..., a_g, b_1, ..., b_g$.

Definition. Let P be an oriented cycle on X. We define closed paths $B(P)$ and $W(P)$ on Γ and Γ^*, respectively, induced by P in the following way.
For an oriented edge $[Q, v]$ of P, we add the black (or white) vertex v to $B(P)$ (or $W(P)$) and the corresponding white (or black) diagonal of $Q \in F(\Lambda)$ to $W(P)$ (or $B(P)$). These closed paths inherit their orientation from P and are homotopic to P. We denote the *one-chains* on X consisting of all black or white edges corresponding to $B(P)$ and $W(P)$ by BP and WP, respectively.

Definition. Let ω be a closed one-form of type $\Diamond$. For $1 \leq k \leq g$, we define its *a_k-period* A_k, its *b_k-period* B_k, its *black a_k-period* A_k^B, its *black b_k-period* B_k^B, its *white a_k-period* A_k^W and its *white b_k-period* B_k^W by

$$A_k := \oint_{\alpha'_k} \omega, \quad A_k^B := 2\int_{B\alpha'_k} \omega, \quad A_k^W := 2\int_{W\alpha'_k} \omega,$$
$$B_k := \oint_{\beta'_k} \omega, \quad B_k^B := 2\int_{B\beta'_k} \omega, \quad B_k^W := 2\int_{W\beta'_k} \omega.$$

Clearly, it hold $2A_k = A_k^B + A_k^W$ and $2B_k = B_k^B + B_k^W$. The periods of ω depend only on the homology classes, as discussed in [BG17], Lemma 5.1.

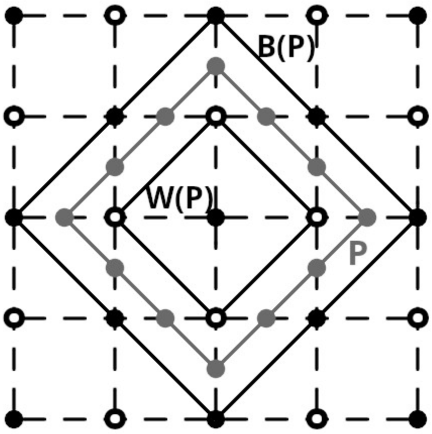

Figure 1.4: Cycles P on X, $B(P)$ on Γ, and $W(P)$ on Γ^*.

We want to integrate discrete one-forms of type $\Diamond$ with given periods to functions on $V(\widetilde{\Lambda})$. This leads us to the following definition.

Definition. Denote the *deck transformation* of $\gamma \in \{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$ by d_γ. We call $f : V(\widetilde{\Lambda}) \to \mathbb{C}$ *multi-valued* with black and white periods $A_1^B, ..., A_g^B, B_1^B, ..., B_g^B, A_1^W, ..., A_g^W, B_1^W, ..., B_g^W \in \mathbb{C}$ if

$$f(d_{\alpha_k} b) = f(b) + A_k^B,\ f(d_{\alpha_k} w) = f(w) + A_k^W,\ f(d_{\beta_k} b) = f(b) + B_k^B,\ f(d_{\beta_k} w) = f(w) + B_k^W$$

hold for any $k = 1, ..., g$ and any vertices $b \in V(\widetilde{\Gamma})$ and $w \in V(\widetilde{\Gamma}^*)$.

The following statement can be found in [BG17], Lemma 5.2.

Lemma 1.10. Let $f : V(\widetilde{\Lambda}) \to \mathbb{C}$ be multi-valued with given black and white periods. Then, *df* defines a closed discrete one-form of type $\Diamond$ on the oriented edges of X with the same black and white periods as f. Conversely, any closed discrete one-form ω can be integrated to a multi-valued function $f : V(\widetilde{\Lambda}) \to \mathbb{C}$, thus it holds $df = \omega$.
If ω is discrete harmonic or discrete holomorphic, then f is as well.

Part (i) of the following proposition gives us the unique solubility of the discrete Dirichlet boundary, and part (ii) guarantees the uniqueness of the basis of holomorphic differentials in the definition below. Proofs can be found in [BG17], Theorem 6.3 and Theorem 6.8.

Proposition 1.11. (i) For any $A_k^B, B_k^B, A_k^W, B_k^W \in \mathbb{C}$, $1 \leq k \leq g$, there exists exactly one discrete harmonic differential ω with these black and white periods.

(ii) For any $A_k^B, A_k^W \in \mathbb{C}$, $1 \leq k \leq g$, there exists exactly one discrete holomorphic differential ω with these black and white a-periods.

(iii) For any $\mathrm{Re}(A_k^B), \mathrm{Re}(B_k^B), \mathrm{Re}(A_k^W), \mathrm{Re}(B_k^W) \in \mathbb{R}$, $1 \leq k \leq g$, there exists exactly one discrete holomorphic differential ω such that its black and white periods have these real parts.

Definition. Let ω_k^B, $1 \leq k \leq g$, be the unique discrete holomorphic differentials with black a-periods $A_j^B = \delta_{jk}$ and vanishing white a-periods. Let $\omega_k^W, k = 1, ..., g$, be the unique discrete holomorphic differentials with white a-periods $A_j^W = \delta_{jk}$ and vanishing black a-periods. We call the basis of these $2g$ differentials the *canonical basis of discrete holomorphic differentials*.

We define the $(g \times g)$-matrices $\Pi^{B,B}, \Pi^{W,B}, \Pi^{B,W}, \Pi^{W,W}$ with entries

$$\Pi^{B,B}_{jk} := 2\int_{Bb_j} \omega^B_k, \quad \Pi^{W,B}_{jk} := 2\int_{Wb_j} \omega^B_k, \quad \Pi^{B,W}_{jk} := 2\int_{Bb_j} \omega^W_k, \quad \Pi^{W,W}_{jk} := 2\int_{Wb_j} \omega^W_k.$$

The *complete discrete period matrix* is the $(2g \times 2g)$-matrix

$$\tilde{\Pi} := \begin{pmatrix} \Pi^{B,W} & \Pi^{B,B} \\ \Pi^{W,W} & \Pi^{W,B} \end{pmatrix}.$$

The *discrete period matrix* is the $(g \times g)$-matrix

$$\Pi := \frac{1}{2}\left(\Pi^{B,W} + \Pi^{B,B} + \Pi^{W,W} + \Pi^{W,B}\right). \tag{1.6}$$

The period matrix (1.6) is the matrix we mostly compare with the continuous period matrix Π_Σ of Σ using the same basis of homology.

Definition. Let ω_k, $1 \leq k \leq g$, be the unique holomorphic differentials with a-periods $A_j = \delta_{jk}$. The $(g \times g)$-matrix Π_Σ - called *continuous period matrix* - is defined by

$$(\Pi_\Sigma)_{j,k} = \int_{b_j} \omega_k, \ 1 \leq j, k \leq g.$$

1.5 Convergence of discrete Dirichlet energies and period matrices

1.5.1 Discretizing functions on Riemann surfaces

Consider a multi-valued function $f : \widetilde{\Sigma} \to \mathbb{C}$ with a- and b-periods $A_1, ..., A_g, B_1, ..., B_g \in \mathbb{C}$. The simplest way to construct a discrete multi-valued function $g : V(\widetilde{\Lambda}) \to \mathbb{C}$ over a discrete Riemann surface $(\widetilde{\Sigma}, \widetilde{\Lambda})$ is to define $g := f|_{V(\widetilde{\Lambda})}$. By definition, g and f have the same periods. In particular, black and white periods of g are the same.

In general, holomorphy does not remain true after such a restriction. For example, consider the holomorphic function $f : \mathbb{C} \to \mathbb{C}, f(x) = x^2$, and let the (orthodiagonal) quadrilateral Q with vertices $b_- = -1, w_- = -i, b_+ = 1, w_+ = 2i$ be part of a discrete Riemann surface. Since

$$f(b_+) - f(b_-) = 0, \quad f(w_+) - f(w_-) = -3,$$

equation (1.2) is not true, and g is not discrete holomorphic (in Q). Later, in the proof of Lemma 2.8, we will see that harmonicity remains also not true.

Proposition 1.11 tells us that we cannot always assume black and white periods being the same. By part (i), we can construct discrete harmonic functions fulfilling the property of equal black and white periods as formulated in Theorem 1.12. But by considering discrete harmonic functions as real and imaginary part of discrete holomorphic functions, we lose this possibility because of part (ii) and part (iii) of Proposition 1.11.
But we can - and so we will, motivated by the discussion above - fix the a-periods along a converging sequence of discrete Riemann surfaces. In addition, if we consider orthodiagonality, we get equality of black and white b-periods in the limit (see Corollary 1.14).

1.5.2 The convergence theorems

To prove the convergence of discrete harmonic functions (Theorem 2.1) and the convergence of discrete holomorphic integrals (Theorem 3.1) to their smooth counterparts in the following chapters, we refer to the main results of [Gün23].
In Theorem 3.19 the convergence of the discrete Dirichlet energies to their smooth counterpart is stated by bounding the differences explicitly by a term tending to zero for the maximal edge length h of the discrete Riemann surfaces tending to zero. Corollary 3.20 bounds the discrete Dirichlet energies explicitly. For our use, we formulate Theorem 3.19 of [Gün23] simplified.

Theorem 1.12. Let $P \in \mathbb{R}^{2g}$ be a given vector of periods and let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a Φ-regular sequence of discrete Riemann surfaces such that their maximal edge lengths converge to zero as $n \to \infty$.
Consider Ω, the unique smooth holomorphic differential with real parts of its a- and b-periods given by P, and ω_n, for each $n \in \mathbb{N}$, the unique discrete holomorphic differentials with real parts of its black and white a- and b-periods given by P such that corresponding black and white periods coincide. Let $U := \int \mathrm{Re}(\Omega)$ and $u_n := \int \mathrm{Re}(\omega_n)$ be the corresponding multi-valued (discrete) harmonic functions. Then, we get

$$\lim_{n\to\infty} E_{\Lambda_n}(u_n) = E_\Sigma(U).$$

Due to the convergence, the sequence $(E_{\Lambda_n}(u_n))_{n\in\mathbb{N}}$ is bounded.

Remark. The statement remains true if we say that the sequence of periods $(P_n)_{n\in\mathbb{N}} \in \mathbb{R}^{2g}$ of $\mathrm{Re}(\omega_n)$ converges to the periods $P \in \mathbb{R}^{2g}$ of $\mathrm{Re}(\Omega)$.

The convergence of the discrete period matrices is formulated in the general case in Theorem 4.3 and in the orthodiagonal case in Corollary 4.4 in [Gün23].

Theorem 1.13. Let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a Φ-regular sequence of discrete Riemann surfaces such that their maximal edge lengths converge to zero for $n \to \infty$ and let $\Pi_n^{B,W}, \Pi_n^{B,B}, \Pi_n^{W,W}, \Pi_n^{W,B}$ be for each $n \in \mathbb{N}$ the blocks of the complete discrete period matrices for a given basis of homology. Then, we have

$$\begin{aligned}\lim_{n\to\infty}\left(\Pi_n^{B,W} + \Pi_n^{B,B}\right) &= \Pi_\Sigma, & \lim_{n\to\infty}\left(\Pi_n^{B,W} - \Pi_n^{W,B}\right) &= 0,\\ \lim_{n\to\infty}\left(\Pi_n^{W,B} + \Pi_n^{W,W}\right) &= \Pi_\Sigma, & \lim_{n\to\infty}\left(\Pi_n^{B,B} - \Pi_n^{W,W}\right) &= 0.\end{aligned}$$

As seen in [Gün23], Lemma 2.6, in the orthodiagonal case the matrices $\Pi_n^{B,W}$ and $\Pi_n^{W,B}$ are purely imaginary and the matrices $\Pi_n^{B,B}$ and $\Pi_n^{W,W}$ are real. This leads to the following Corollary.

Corollary 1.14. Let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a Φ-regular sequence of orthodiagonal discrete Riemann surfaces such that their maximal edge lengths converge to zero for $n \to \infty$, and let $\Pi_n^{B,W}, \Pi_n^{B,B}, \Pi_n^{W,W}, \Pi_n^{W,B}$, $n \in \mathbb{N}$, be as in Theorem 1.13. Then, we get

$$\begin{aligned}\lim_{n\to\infty}\Pi_n^{B,W} &= i\mathrm{Im}(\Pi_\Sigma) = \lim_{n\to\infty}\Pi_n^{W,B},\\ \lim_{n\to\infty}\Pi_n^{B,B} &= \mathrm{Re}(\Pi_\Sigma) = \lim_{n\to\infty}\Pi_n^{W,W}.\end{aligned}$$

Chapter 2

Convergence of discrete harmonic functions

The objective of this chapter is the proof of the convergence of a sequence of discrete harmonic functions over a nondegenerate uniform sequence of orthodiagonal discrete Riemann surfaces - with maximal edge length converging to zero - to the harmonic function on the smooth Riemann surface with a given converging sequence of periods. The orthodiagonality will be important because of the orthodiagonal Green's identity proven in Corollary 1.9 and the Maximum principle for discrete harmonic functions on bounded orthodiagonal surfaces (see Lemma 2.2).

Theorem 2.1. Let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a nondegenerate uniform sequence of orthodiagonal discrete Riemann surfaces such that the maximal edge length converges to zero as $n \to \infty$. Let $(P_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{2g}$ be a sequence of vectors converging to a vector $P \in \mathbb{R}^{2g}$. Let $(p_n, p_n') \in E(\Lambda_n)$, $n \in \mathbb{N}$, be a sequence of edges such that $p_n \to p \in \Sigma$ as $n \to \infty$.
Let $u : \widetilde{\Sigma} \to \mathbb{R}$ be the unique multi-valued harmonic function with periods given by P that satisfies $u(p) = 0$ and, for any $n \in \mathbb{N}$, let $v_n : V(\widetilde{\Lambda}_n) \to \mathbb{R}$ be the unique real multi-valued discrete harmonic function with equal black and white periods given by P_n that satisfies $v_n(p_n) = v_n(\underset{\sim}{p_n'}) = 0$. Then, the harmonic sequence $(v_n)_{n\in\mathbb{N}}$ converges to u uniformly on each compact subset $\Omega \subseteq \widetilde{\Sigma}$, i.e.,

$$\lim_{n\to\infty} \max_{z\in V(\widetilde{\Lambda}_n)\cap\Omega} |v_n(z) - u(z)| = 0.$$

To prove this, we will orientate ourselves on [BS16], Chapter 5, and [Sko13]. Moreover we will need the convergence of the corresponding Dirichlet energies stated in Theorem 1.12 and proven in [Gün23].

Remark. The statement remains true if we consider more generally different black and white b-periods for the discrete harmonic function converging separately to the b-periods of the smooth harmonic function. We will see this in the proof in Section 2.3.

2.1 Equicontinuity on orthodiagonal discrete cones

Initially we consider simply-connected subsets of discrete Riemann surfaces containing at most one conical singularity. Our goal is to bound the difference of the images of two same-coloured points under a discrete harmonic function by a term including the discrete Dirichlet energy. This is formulated in the Equicontinuity Lemma (see Lemma 2.5).
To achieve this, we need the maximum principle for discrete harmonic functions, which only works for orthodiagonal discrete surfaces.

Let (Σ, Λ) be a Φ-regular discrete Riemann surface without boundary with maximal edge length h.

Definition. A *discrete cone* (around $O \in V(\Lambda)$) is a discrete Riemann surface (Σ_O, Λ_O) with boundary such that $\Sigma_O \subseteq \Sigma$ is the simply connected union of all quadrilaterals $Q \in F(\Lambda_O) \subseteq F(\Lambda)$ inside an given intrinsic cone D_O with apex O containing $\text{star}(O)$ (by definition of D_O it is at most $O \in V(\Lambda_O)$ a conical singularity of Σ). We call $\partial\Lambda_O := V(\Lambda_O) \cap \partial\Sigma_O$ *boundary* of the discrete cone.

The *discrete Dirichlet energy* of $u : V(\Lambda_O) \to \mathbb{R}$ is given by

$$E_{\Lambda_O}(u) := \sum_{Q \in F(\Lambda_O)} |\nabla_Q u|^2 \text{area}(Q).$$

The *eccentricity* $\mathcal{E}$ of the discrete cone is given by the infimum of the numbers Const such that the following conditions hold.

- For each quadrilateral $Q \in F(\Lambda_O)$ with black vertices b_+, b_- and white vertices w_+, w_- we have
$$\frac{1}{\text{Const}} < \frac{|b_+ - b_-|}{|w_+ - w_-|} < \text{Const}, \quad \frac{1}{\text{Const}} < \sin(\varphi_Q) \leq 1. \tag{2.1}$$
- The number of vertices in an arbitrary disk of radius h is less than Const.

Remark. The eccentricity $\mathcal{E}$ is also well-defined over a nondegenerate uniform sequence of discrete Riemann surfaces because of (1.1).

Lemma 2.2 (Maximum principle)**.** Let (Σ_O, Λ_O) be an orthodiagonal discrete cone with apex $O \in V(\Lambda)$ and let $u : V(\Lambda_O) \to \mathbb{R}$ be a discrete harmonic function. Then, we have

$$\max_{z \in V(\Gamma_O)} u(z) = \max_{w \in \partial\Gamma_O} u(w), \quad \max_{z \in V(\Gamma^*_O)} u(z) = \max_{w \in \partial\Gamma^*_O} u(w),$$

where $\partial\Gamma_O := V(\Gamma_O) \cap \partial\Sigma_O$ and $\partial\Gamma^*_O := V(\Gamma^*_O) \cap \partial\Sigma_O$. In particular, it holds

$$\max_{z \in V(\Lambda_O)} u(z) = \max_{w \in \partial\Lambda_O} u(w).$$

Proof. Using the harmonicity of u and the orthodiagonality of (Σ_O, Λ_O), we get

$$0 = \sum_{Q_s \in \text{star}(v)} c_s(u(v_s) - u(v)), \ c_s := \frac{|\rho_s|^2}{\text{Re}(\rho_s)} > 0$$

for any $v \in V(\Lambda_O) \setminus \partial\Lambda_O$ because of Proposition 1.1 and Proposition 1.5, and therefore

$$u(v) = \frac{\sum_{Q_s \in \text{star}(v)} c_s u(v_s)}{\sum_{Q_s \in \text{star}(v)} c_s}. \tag{2.2}$$

Assume that such an inner vertex v is a maximizer of u. Because equation (2.2) deals with the vertices of $\text{star}(v)$ of the same colour as v, these vertices are also maximizer of u. Therefore, u is constant on that colour. The same argumentation fits for the other colour. Hence, the statement fits also for $V(\Lambda_O)$. ☐

Remark. By using minima in the lemma above and talking about minimizer in the proof, instead, we get the so-called *Minimum principle*.

The maximum principle does not hold for arbitrary non-orthodiagonal lattices. The following example is inspired by [Sko13], Example 3.6.

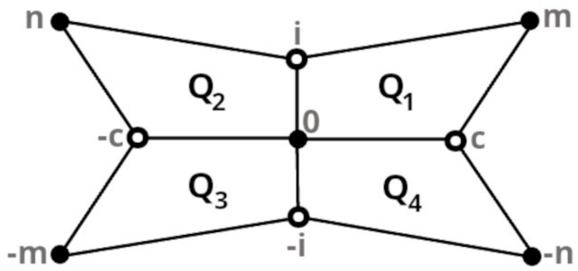

Figure 2.1: The non-orthodiagonal lattice Λ_M, $M > 1$, with $m := \sqrt{2}M(c+i)$, $n := \sqrt{2}M(-c+i)$.

Example. For $M > 1$ let Λ_M be the lattice formed by the non-orthodiagonal quadrilateral with black vertices $0, \sqrt{2}M(c+i)$ and white vertices c, i and the three quadrilaterals obtained by reflecting at the coordinate axes and the origin (see Figure 2.1), where $c = \cot(\frac{\pi}{8})$ is the positive zero of the polynomial

$$q = (2-\sqrt{2})t^2 - (2+\sqrt{2}). \tag{2.3}$$

Define

$$f : V(\Lambda_M) \to \mathbb{R}, \ f(z) := \begin{cases} M, & \text{if } z = 0, \\ 1, & \text{if } z = \pm i, \\ 0, & \text{else.} \end{cases}$$

To prove that f is harmonic we verify $\Delta f(0) = 0$ with the help of Proposition 1.5. Therefore, we investigate $\rho_s := \rho_{Q_s}, s = 1, ..., 4$, as defined in Figure 2.1. Calculations lead to

$$\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2) = \mathrm{Re}(\rho_3) = \mathrm{Re}(\rho_4) = \frac{2c}{\sqrt{2}M(c^2+1)},$$
$$\mathrm{Im}(\rho_1) = \mathrm{Im}(\rho_3) = \frac{c^2-1}{\sqrt{2}M(c^2+1)}, \quad \mathrm{Im}(\rho_2) = \mathrm{Im}(\rho_4) = \frac{1-c^2}{\sqrt{2}M(c^2+1)},$$
$$|\rho_1|^2 = |\rho_2|^2 = |\rho_3|^2 = |\rho_4|^2 = \frac{1}{2M^2}.$$

In particular, the lattice is actually non-orthodiagonal. By Proposition 1.5, $\Delta f(0)$ is proportional to

$$\sum_{Q_s \in \mathrm{star}(0)} \left(|\rho_1|^2 (f(v_s) - f(0)) + \mathrm{Im}(\rho_s)(f(v_s') - f(v_{s-1}')) \right) = -\frac{2}{M} + \frac{2}{M} \cdot \frac{2c^2-2}{\sqrt{2}(c^2+1)},$$

which is proportional to

$$-1 + \frac{2c^2-2}{\sqrt{2}(c^2+1)} = \frac{(2-\sqrt{2})c^2 - (2+\sqrt{2})}{\sqrt{2}(c^2+1)}.$$

Hence, $\Delta f(0) = 0$ because c is zero of the polynomial q given in (2.3). Therefore, f is discrete harmonic, because $0 \in V(\Lambda_M)$ is the only non-boundary vertex. Further, f does not fulfill the Maximum Principle.

Lemma 2.3 (Path Energy Lemma)**.** Let (Σ_O, Λ_O) be an orthodiagonal discrete cone, $u : V(\Lambda_O) \to \mathbb{R}$ be a discrete harmonic function and $w_0 w_1 ... w_m \subset \Gamma_O$ (or Γ_O^*) be a path, i.e., $w_0w_1, w_1w_2, ..., w_{m-1}w_m$ are all black (or white) diagonals of Λ_O. The *path energy* of u along $w_0w_1...w_m$ is given by

$$E_{w_0...w_m}(u) := \sum_{\substack{Q \in F(\Lambda_O):\ \exists i \in \{1,...,m\}: \\ w_{i-1}w_i \text{ is diagonal of } Q}} |\nabla_Q u|^2 \cdot \mathrm{area}(Q).$$

Then, the path energy fulfills

$$E_{w_0\ldots w_m}(u) \geq \mathrm{Const}\frac{(u(w_m)-u(w_0))^2}{m\mathcal{E}}.$$

Proof. W.l.o.g. let the path be black. By using the definitions of the discrete gradient and of the eccentricity $\mathcal{E}$ we get

$$\begin{aligned}
E_{w_0\ldots w_m}(u) &= \sum_{\substack{Q\in F(\Lambda_O):\ \exists i\in\{1,\ldots,m\}: \\ w_{i-1}w_i \text{ is diagonal of } Q}} \frac{(u(b_+)-u(b_-))^2}{|b_+b_-|^2}\cdot\frac{1}{2}|b_+b_-||w_+w_-| \\
&= \sum_{\substack{Q\in F(\Lambda_O):\ \exists i\in\{1,\ldots,m\}: \\ w_{i-1}w_i \text{ is diagonal of } Q}} \frac{(u(b_+)-u(b_-))^2}{2}\cdot\frac{|w_+w_-|}{|b_+b_-|} \\
&\overset{(2.1)}{\geq} \sum_{\substack{Q\in F(\Lambda_O):\ \exists i\in\{1,\ldots,m\}: \\ w_{i-1}w_i \text{ is diagonal of } Q}} \frac{(u(b_+)-u(b_-))^2}{2}\cdot\frac{1}{\mathcal{E}} \\
&= \mathrm{Const}\cdot\sum_{k=1}^{m}\frac{(u(w_k)-u(w_{k-1}))^2}{\mathcal{E}} \\
&\geq \mathrm{Const}\cdot\frac{(u(w_m)-u(w_0))^2}{m\mathcal{E}}.
\end{aligned}$$

To get the last inequality, note the fact that $2\alpha\beta \leq \alpha^2+\beta^2$ holds for any $\alpha,\beta\in\mathbb{R}$. We conclude the proof with

$$\left(\sum_{k=1}^{m}\beta_k\right)^2 = \sum_{k=1}^{m}\beta_k^2 + \sum_{k>l}2\beta_k\beta_l \leq m\cdot\sum_{k=1}^{m}\beta_k^2,$$

where $\beta_k := u(w_k)-u(w_{k-1}),\ k=1,\ldots,m.$ ☐

Remark. In the non-orthodiagonal case we get by using $\mathrm{area}(Q) = \frac{1}{2}|b_+-b_-||w_+-w_-|\cdot\sin(\varphi_Q)$ and $\sin(\varphi_Q) \geq \frac{1}{\mathcal{E}}$ for each $Q\in F(\Lambda_O)$ in Lemma 2.3 the estimate

$$E_{w_0\ldots w_m}(u) \geq \mathrm{Const}\frac{(u(w_m)-u(w_0))^2}{m\mathcal{E}^2},$$

as seen in [Sko13], Lemma 4.1.

Lemma 2.4 (Rectangle Capacity Lemma)**.** A rectangle $y\times 2h$ with $y>2h$ contains at most $\mathrm{Const}\cdot\frac{\mathcal{E}y}{2h}$ vertices of Γ (or Γ^*, respectively).

Proof. We can cover the rectangle by $\mathrm{Const}\cdot\lfloor y/2h\rfloor$ discs of radius h. Then, by definition of the eccentricity $\mathcal{E}$ there are at most $\mathrm{Const}\cdot\frac{\mathcal{E}y}{2h}$ vertices in the rectangle. ☐

Lemma 2.5 (Equicontinuity Lemma)**.** For $O\in V(\Lambda)$ let (Σ_O,Λ_O) be an orthodiagonal discrete cone with singularity index γ_O bounded by a piecewise-geodesic broken line. Let $u: V(\Lambda_O)\to\mathbb{R}$ be a discrete harmonic function and let $z,w\in\Gamma_O$ (or $z,w\in\Gamma_O^*$) be at the distance $|zw|\geq 2h$. Define

$$r := \mathrm{dist}(O,\partial\Sigma_O) - \max\{|Oz|,|Ow|\} - 2h,\ \gamma_O' := \max\{\gamma_O^2, 1/\gamma_O^2\}.$$

Assume that $r > 3\gamma_O'|zw|$. Then there is a constant $\mathrm{Const}_{\mathcal{E},\gamma_O}$ such that

$$|u(z)-u(w)| \leq \mathrm{Const}_{\mathcal{E},\gamma_O}\cdot E_{\Lambda_O}(u)^{1/2}\cdot\ln\left(\frac{r}{3\gamma_O'|zw|}\right)^{-1/2}.$$

For $|zw| < 2h < r/3\gamma_O'$ the same inequality holds with $|zw|$ replaced by $2h$.

The first part of the proof follows the proof of [Sko13], Lemma 2.4, without considering separately the case of square lattices. The second part follows the proof of [BS16], Lemma 5.1, which also regards to [Sko13]. In our context, part (I) of the proof considers flat discrete cones without a conical singularity as apex. Part (II) of the proof generalizes to arbitrary orthodiagonal discrete cones we will rearrange into flat cones.

Let $y \in \Sigma_O$ is the point in the middle of the direct intrinsic line between z and w. It holds

$$r < \mathrm{dist}(y, \partial\Sigma_O) \leq \mathrm{dist}(y, \partial\Lambda_O). \tag{2.4}$$

Assume the first inequality does not hold. Then we get

$$\begin{aligned}\mathrm{dist}(O, \partial\Sigma_O) &\leq |Oy| + \mathrm{dist}(y, \partial\Sigma_O) \leq |Oy| + r = \mathrm{dist}(O, \partial\Sigma_O) - \max\{|Oz|, |Ow|\} + |Oy| - 2h\\ &\leq \mathrm{dist}(O, \partial\Sigma_O) - 2h < \mathrm{dist}(O, \partial\Sigma_O),\end{aligned}$$

which is a contradiction. The relation $\mathrm{dist}(y, \partial\Sigma_O) \leq \mathrm{dist}(y, \partial\Lambda_O)$ is clear since $\partial\Lambda_O \subseteq \partial\Sigma_O$.

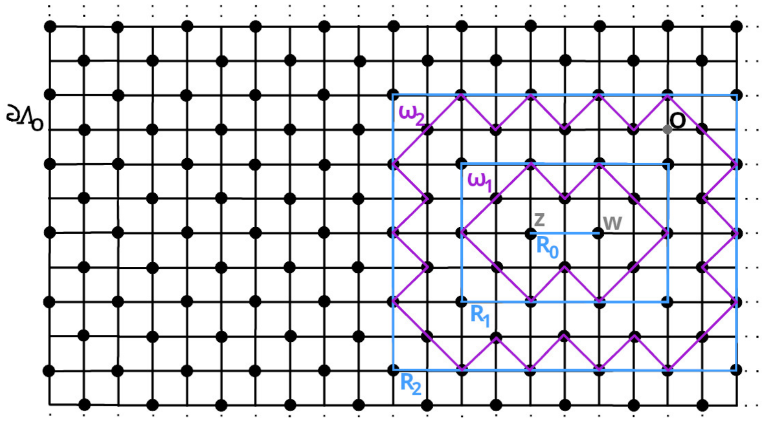

Figure 2.2: Discrete cone around O with $r = 18h - 5h - 2h = 11h > 6h = 3\gamma'_O|zw|$ because of $|zw| = 2h$ and $\gamma'_O = 1$. It is $\lfloor \frac{r-|zw|}{2h'} \rfloor = \lfloor \frac{11h-2h}{4h} \rfloor = 2$.

Proof of Lemma 2.5. W.l.o.g. let $z, w \in \Gamma_O$ and $u(z) > u(w)$. Define $h' := 2h$.[1] We consider two cases.

(I) Let $\gamma_O = \gamma'_O = 1$ (i.e., O is not a conical singularity). Therefore, we assume $\Sigma_O \subset \mathbb{C}$. Define closed rectangles

$$R_m,\ m \in \left\{0, 1, \ldots, \left\lfloor \frac{r - |zw|}{2h'} \right\rfloor \right\},$$

of size $2mh' \times (2mh' + |zw|)$ centered in $\frac{z+w}{2}$ such that the side $2mh'$ is orthogonal to the segment zw (R_0 is just the straight line connecting z and w). It is $R_m \subseteq \Sigma_O$ for each valid m, because each rectangle lies inside a circle of radius r around the midpoint between z and w. This is true, because half of the diameter of the biggest R_m is bounded by

$$\sqrt{\left(\frac{|zw|}{2} + \frac{r - |zw|}{2}\right)^2 + \left(\frac{r - |zw|}{2}\right)^2} = \frac{1}{2}\sqrt{r^2 + (r - |zw|)^2} \leq \frac{1}{2}\sqrt{r^2 + r^2} < r.$$

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[1] In [Sko13] h is defined to be twice the maximal edge length of the quad-graph and in [BS16], Chapter 5, h' is used in a similar way. We use this definition for the comparability.

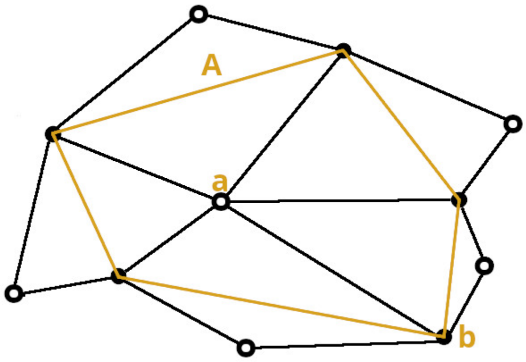

Figure 2.3: Illustration of $A \in F(\Gamma_O)$ via a star.

The diameter of each bounded face $A \in F(\Gamma_O)$ is at most $2h$. To see this, let $a \in V(\Gamma_O^*)$ be the unique white vertex within A. Further, let b be a vertex of A with maximal distance to a (see Figure 2.3). By assumption, we get

$$2h \geq 2|ba| \geq \mathrm{diam}(A). \tag{2.5}$$

Therefore, there is a simply closed path $\omega_m \subset \Gamma_O$ within R_m around R_{m-1} for each valid $m \geq 1$ because of $\mathrm{dist}(\partial R_m, \partial R_{m-1}) = 2h$ (see Figure 2.2, for example).
The area between the boundaries of the rectangles R_m of size $4mh \times (4mh + |zw|)$ and R_{m-1} of size $4(m-1)h \times (4(m-1)h + |zw|)$ is covered by two rectangles of size $(4mh + |zw|) \times 2h$ and two rectangles of size $(4(m-1)h) \times 2h$. By Lemma 2.4, this area contains less than

$$\begin{aligned} M_m &:= 2 \cdot \mathrm{Const} \cdot \mathcal{E}\left(4m + \frac{|zw|}{2h}\right) \\ &\geq 2 \cdot \mathrm{Const} \cdot \frac{\mathcal{E} \cdot (4mh + |zw|)}{2h} + 2 \cdot \mathrm{Const} \cdot \frac{\mathcal{E} \cdot (4(m-1)h)}{2h} > 0 \end{aligned}$$

vertices of Γ_O. Therefore, the number of edges in the path $\omega_m \subset \Gamma_O$ is less than this number, too.

For $m = 1, \ldots, \lfloor \frac{r-|zw|}{2h'} \rfloor$ let $\Gamma_m \subset \Gamma_O$ be the sub-graph of all vertices bounded and included by ω_m. Define $\delta := |u(z) - u(w)| > 0$. For each $m = 1, \ldots, \lfloor \frac{r-|zw|}{2h'} \rfloor$ we get, by using Lemma 2.2, the maximizer z_m and minimizer w_m of u on Γ_m at $\partial\Gamma_m$. Clearly, we get

$$|u(z_m) - u(w_m)| \geq |u(z) - u(w)| = \delta > 0.$$

Let us estimate the Energy $E_{\Lambda_O}(u)$. The number of vertices of the shortest path $\psi_m \subseteq \omega_m$ from z_m to w_m in $\partial\Gamma_m$ is obviously bounded by M_m. By Lemma 2.3 we get

$$\begin{aligned} E_{\psi_m}(u) &\geq \mathrm{Const}\frac{(u(z_m) - u(w_m))^2}{\mathcal{E} M_m} \\ &\geq \mathrm{Const}\frac{\delta^2}{\mathcal{E}} \cdot \frac{1}{M_m} = \mathrm{Const}\frac{\delta^2}{\mathcal{E}} \cdot \frac{1}{\mathrm{Const} \cdot \mathcal{E}\left(4m + \frac{|zw|}{2h}\right)} \\ &= \mathrm{Const}\frac{\delta^2}{\mathcal{E}^2} \cdot \frac{h'}{4mh' + |zw|}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} E_{\Lambda_O}(u) &\geq \sum_{m=1}^{\lfloor \frac{r-|zw|}{2h'} \rfloor} E_{\psi_m}(u) \geq \mathrm{Const}\frac{\delta^2}{\mathcal{E}^2} \sum_{m=1}^{\lfloor \frac{r-|zw|}{2h'} \rfloor} \frac{h'}{4mh'+|zw|} \\ &\geq \mathrm{Const}\frac{\delta^2}{\mathcal{E}^2} \int_{h'}^{\frac{r-|zw|}{2}} \frac{dt}{4t+|zw|} \\ &= \mathrm{Const}\frac{\delta^2}{\mathcal{E}^2} \cdot \frac{1}{4} \ln\left(\frac{2r-|zw|}{4h'+|zw|}\right). \end{aligned}$$

By using $|zw| < r/3$ we get

$$\frac{2r-|zw|}{4h'+|zw|} \geq \frac{\frac{5}{3}r}{4h'+|zw|} = \frac{r}{\frac{12}{5}h' + \frac{3}{5}|zw|},$$

and so

$$E_{\Lambda_O}(u) \geq \mathrm{Const}\frac{\delta^2}{\mathcal{E}^2} \cdot \ln\left(\frac{r}{\frac{12}{5}h' + \frac{3}{5}|zw|}\right) \geq \mathrm{Const}\frac{\delta^2}{\mathcal{E}^2} \cdot \ln\left(\frac{r}{3\max\{|zw|, h'\}}\right).$$

Hence, we conclude

$$|u(z) - w(z)| = \delta \leq \sqrt{\frac{\mathcal{E}^2}{\mathrm{Const}} E_{\Lambda_O}(u) \ln\left(\frac{r}{3\max\{|zw|, h'\}}\right)^{-1}}.$$

(II) Now consider $\gamma_O \neq 1$. Let the map $q : \Sigma_O \to \mathbb{R}^2$ be given in polar coordinates by

$$q : (\rho, \phi) \mapsto \left(\sqrt{\gamma_O'}\rho, \gamma_O \phi\right).$$

We identify each face of Λ_O (respectively, its image under q) with a quadrilateral in $\mathbb{C}$ by an orientation-preserving isometry (respectively, its image under q), as usual. This makes $q(\Lambda_O)$ an orthodiagonal quad-surfaced polygon (we do not consider the geometry directly on $q(\Sigma_O)$) and $u \circ q^{-1} : q(\Lambda_O) \to \mathbb{R}$ a discrete harmonic function.
The map q increases distances by a factor at most γ_O'. This is because of the radial increase of $\sqrt{\gamma_O'}$ and angle increase of $\sqrt{\gamma_O'}$ or angle decrease of $\frac{1}{\sqrt{\gamma_O'}}$, respectively. Therefore $q(\Lambda_O)$ has (in $\mathbb{R}^2$) a maximal edge length of at most $\gamma_O' h$ and an eccentricity of at most $\mathrm{Const}_{\gamma_O, \mathcal{E}}$.
Let R be the square of side length r with center $x := \frac{q(z)+q(w)}{2}$ and the sides parallel and orthogonal to the segment $q(z)q(w)$. By using (2.4) we get

$$r < \mathrm{dist}(q^{-1}(x), \partial\Lambda_O) \leq \mathrm{dist}(x, q(\partial\Lambda_O)). \tag{2.6}$$

The second inequality holds because of the radial increase through q and the construction of a discrete cone being maximal inside an intrinsic circle. Therefore, potential angle decrease is compensated by the radial increase.
Therefore, we have $R \subseteq q(\Sigma_O)$. As in (I) we can construct rectangles of side length at most r being in $q(\Sigma_O)$. Since $r > 3\gamma_O'|zw| \geq 3|q(z) - q(w)|$, the statement follows by part (I).

□

2.2 Harmonicity of a uniform limit

In this section we consider quad-surfaced polygons in $\mathbb{C}$, i.e., flat simply connected quad-graphs. We will treat them as sub-graphs of a discrete Riemann surface with maximal edge length h and eccentricity $\mathcal{E}$. Let $R \subseteq \mathbb{C}$ be a square inside such a given polygon with edge length $r > 0$.
Our goal is to prove in Lemma 2.10 that the continuous function on the polygon, we get as the uniformly limit of the restriction of a sequence of discrete harmonic functions to the black (or white) vertices, is harmonic, too. Because of the orthodiagonal discrete Green's identity (Corollary 1.9) we need to require orthodiagonality there.
As preparation we need Lemma 2.8 which defines and proves an approximation of the discrete Laplacian to the classical one. This statement will be shown for general quadrilaterals. The following two lemmata are a preparations for the Laplacian Approximation Lemma.

Lemma 2.6. There are at most Const $\cdot \frac{\mathcal{E}r}{h}$ faces with vertices b_-, w_-, b_+, w_+ such that $b_- \in V(\Gamma) \cap R$ and $b_+ \notin R$.

Proof. We have already seen in (2.5) that the diameter of each face of Γ is at most $2h$. Therefore, all vertices of the considered faces are in the $2h$-neighborhood of ∂R. By Lemma 2.4 the number of these vertices is at most Const $\cdot \frac{\mathcal{E}r}{h}$. So, the number of faces is at most Const $\cdot \frac{\mathcal{E}r}{h}$, too. ☐

The following version of the Gradient Approximation Lemma can be found in [Sko13], Lemma 4.5. We use this version, because we consider at this moment only quad-surfaced polygons. A more general version which respects also conical singularities can be found in [Gün23], Lemma 3.4.

Lemma 2.7 (Gradient Approximation)**.** Let $G : \mathbb{C} \to \mathbb{R}$ be a smooth function, (Σ, Λ) a quad-surfaced polygon with maximal edge length h and $g := G|_{V(\Lambda)}$. Then, for each $z \in \Sigma$, we have

$$|\nabla G(z) - \nabla_Q g| \leq \text{Const} \cdot \mathcal{E}h \max_{z \in \text{Conv}(\Sigma)} |D^2 G(z)|,$$

where $Q \in F(\Lambda)$ is covering z, Conv(Σ) describes the convex hull of Σ, and

$$|D^k G(z)| := \max_{0 \leq j \leq k} \left| \frac{\partial^k G}{\partial^j x \partial^{k-j} y}(z) \right|.$$

At next, we formulate and prove the Laplacian Approximation Lemma. We will be guided by [Sko13], Lemma 2.5. Note that we need twice the discrete Laplacian because of variation of our theory.

Lemma 2.8 (Laplacian Approximation)**.** Let $G : \mathbb{C} \to \mathbb{R}$ be a smooth function, (Σ, Λ) a quad-surfaced polygon with maximal edge length h, $g := G|_{V(\Lambda)}$ and $R \subseteq \Sigma \setminus \partial\Sigma$ a square of side length $r > 2h$. Then there exists a positive constant Const$_{\mathcal{E}}$, where $\mathcal{E}$ is the eccentricity of (Σ, Λ), such that

$$\left| \sum_{z \in R \cap V(\Gamma)} 2\Delta g(z) - \int_R \Delta_\Sigma G dx dy \right| \leq \text{Const}_{\mathcal{E}} \left(2hr \max_{z \in R} |D^2 G(z)| + r^3 \max_{z \in R} |D^3 G(z)| \right). \tag{2.7}$$

The same inequality holds for Γ^* instead of Γ.

Proof. W.l.o.g. let the center of R be the origin. By the *Taylor formula* we get

$$G(z) = a_0 + a_1 \text{Re}(z) + a_2 \text{Im}(z) + a_3 |z|^2 + a_4 \text{Re}(z^2) + a_5 \text{Im}(z^2) + \tilde{G}(z),$$

where $a_0, ..., a_5 \in \mathbb{R}$ and $\tilde{G} : \mathbb{C} \to \mathbb{R}$ is a function such that $D^k \tilde{G}(0) = 0$ for $k = 0, 1, 2$.
To prove the whole statement we prove it for the seven summands in the Taylor formula above.

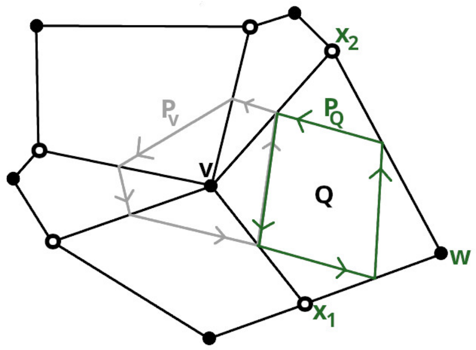

Figure 2.4: Notation of the proof.

1. Let $G(z) = 1$. Obviously then are $\Delta_\Sigma G = 0$ and $\Delta g = 0$.
2. Let $G(z) = \mathrm{Re}(z)$. As discussed in Chapter 1.3, g is discrete harmonic. Furthermore, G is harmonic. This means that the left-hand side of inequality (2.7) is zero and the statement holds.
3. Let $G(z) = \mathrm{Im}(z)$. This case is treated in the same way as the second case.
4. Let $G(z) = |z|^2$. Let $v \in V(\Lambda) \cap R$. We calculate the discrete Laplacian of g at v. Firstly, we get $\Delta g = \Delta g_v$ for $g_v : V(\Lambda) \to \mathbb{C}$, $g_v(z) := |z - v|^2$ because of

   $$|z|^2 = |z - v|^2 + 2\mathrm{Re}(\bar{v}z) - |v|^2$$

   and the first cases. The definition of the discrete Hodge star and Theorem 1.2 yields

   $$\Delta g(v) = \Delta g_v(v) = \star d \star dg_v(v) = d \star dg(F_v) = \oint_{P_v} \star dg_v = \sum_{Q \sim v} \int_{[Q,v]} \star dg_v.$$

   Let $Q \in F(\Lambda)$ be incident to v. As pictured in Figure 2.4 denote its vertices in counterclockwise order by v, x_1, w, x_2. Note that P_Q is reversed oriented relating to P_v. Further, let z' be the intersection point of the bisectors of the diagonals of Q (see Figure 2.5). Denote a function

   $$k : V(\Lambda) \to \mathbb{R},\ k(z) := \mathrm{Re}(\overline{(z' - v)}z).$$

   Again, we split

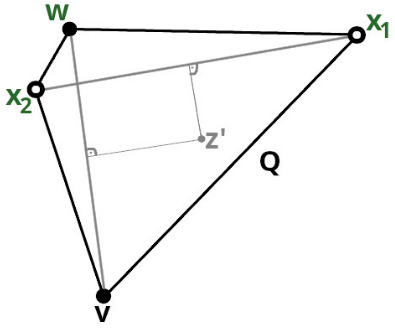

Figure 2.5: Construction of z'.

$$g_v(z) = |z - z' + z' - v|^2 = |z - z'|^2 + |z' - v|^2 + 2\mathrm{Re}(\overline{(z' - v)}z) - 2\mathrm{Re}(\overline{(z' - v)}z')$$

and get $dg_v = 2dk$ in Q because of $|w - z'| = |v - z'|$ and $|x_2 - z'| = |x_1 - z'|$ and vanishing constant terms. We calculate

$$\begin{aligned}\int_{[Q,v]} \star dg_v &= 2\int_{[Q,v]} \star dk = 2\left(-i\partial_\Lambda k(Q)\int_{[Q,v]} dz + i\bar{\partial}_\Lambda k(Q)\int_{[Q,v]} d\bar{z}\right)\\ &= \frac{1}{\mathrm{area}(Q)}\left(\sum_{x\sim Q} k(x)([Q,v]\overline{[Q,x]} + \overline{[Q,v]}[Q,x])\right).\end{aligned}$$

At this point note again the differ of orientation of P_v and P_Q. In particular, $[Q,x]$ for $x = v$ is defined as $-[Q,v]$. This, and using $2[Q,v] = x_2 - x_1$ and $2[Q,x_2] = -2[Q,x_1] = v - w$ leads to

$$\begin{aligned}\int_{[Q,v]} \star\, dg_v &= \frac{2}{\mathrm{area}(Q)}\Big((k(w) - k(v))|[Q,v]|^2 + (k(x_2) - k(x_1))\mathrm{Re}([Q,v]\overline{[Q,x_2]})\Big)\\ &= \frac{1}{2\mathrm{area}(Q)}\Big(\mathrm{Re}(\overline{(z' - v)}(w - v))|x_2 - x_1|^2 + \mathrm{Re}(\overline{(z' - v)}(x_2 - x_1))\mathrm{Re}((x_2 - x_1)\overline{(v - w)})\Big)\\ &= \frac{1}{2\mathrm{area}(Q)}\Big(\mathrm{Re}(\overline{(z' - v)}(x_2 - x_1)\overline{(x_2 - x_1)}(w - v)) - \mathrm{Re}(\overline{(z' - v)}(x_2 - x_1))\mathrm{Re}(\overline{(x_2 - x_1)}(w - v))\Big).\end{aligned}$$

Let $\alpha, \beta \in \mathbb{C}$. It holds $\mathrm{Re}(\alpha\beta) = \mathrm{Re}(\alpha)\mathrm{Re}(\beta) - \mathrm{Im}(\alpha)\mathrm{Im}(\beta)$, and also $\mathrm{Im}(\alpha\bar{\beta}) = -\mathrm{Im}(\bar{\alpha}\beta)$. This observation leads to

$$\int_{[Q,v]} \star dg_v = \frac{1}{2\mathrm{area}(Q)}\Big(\mathrm{Im}(\overline{(z' - v)}(x_2 - x_1))\mathrm{Im}(\overline{(w - v)}(x_2 - x_1))\Big),$$

where $\mathrm{Im}(\overline{(w - v)}(x_2 - x_1))$ equals $2\mathrm{area}(Q)$. Summarized, we get

$$2\Delta g(v) = \sum_{Q\sim v} 4\mathrm{area}(vx_1z'x_2),$$

where the vertices x_1, z', x_2 depend on Q. The area of a closed broken line is understood in oriented sense. This fits with the classical Laplacian where we get

$$\int_R \Delta_\Sigma G dxdy = \int_R \Delta_\Sigma(x^2 + y^2)dxdy = 4\mathrm{area}(R).$$

Now, we estimate the left-hand side of inequality (2.7). For simplicity we assume that each quadrilateral of Λ is convex (otherwise replace the diagonals x_1x_2 in the argument below with two edges of the non convex quadrilaterals).

At first, let R_Λ be the union of oriented diagonals $\overrightarrow{x_1x_2}$ such that the faces $vx_1wx_2 \in F(\Lambda)$ fulfill $v \in R \cap V(\Gamma)$ and $w \notin R$ (see Figure 2.6). Because R_Λ is contained in the h-neighborhood of the curve ∂R, we get

$$|\mathrm{area}(R_\Lambda) - \mathrm{area}(R)| \leq 2rh + h^2 < 4hr \tag{2.8}$$

since $2h < r$.
At second, the sine theorem for the triangle with the vertices $z', z_1 := \frac{v+w}{2}, z_2 := \frac{x_1+x_2}{2}$ yields

$$|z' - z_2| \leq \frac{|z_1 - z_2|}{\sin\measuredangle z'} = \frac{1}{\sin\measuredangle(vw, x_1x_2)}\cdot\left|\frac{v - x_1}{2} + \frac{w - x_2}{2}\right| \leq \mathrm{Const}\cdot\mathcal{E}h$$

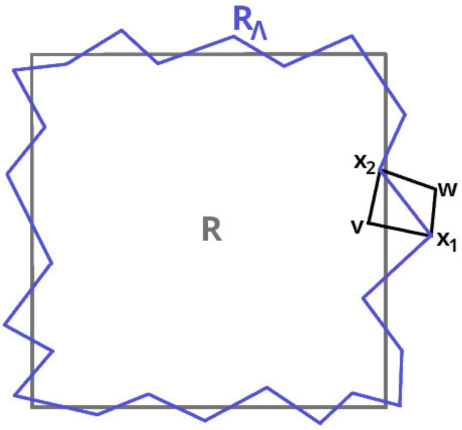

Figure 2.6: Construction of R_Λ.

because of the Φ-regularity of Λ. Hence, we get

$$|\mathrm{area}(z'x_1x_2)| \leq h \cdot |z' - z_2| \leq \mathrm{Const} \cdot \mathcal{E}h^2. \tag{2.9}$$

Lemma 2.6 tells us that there are at most Const $\cdot \frac{\mathcal{E}r}{h}$ faces with with black vertices v, w such that $v \in R$ and $w \notin R$. Therefore, the calculated Laplacian of g and G and the estimates (2.8) and (2.9) conclude

$$\begin{aligned}\left|\sum_{z\in R\cap V(\Gamma)} 2\Delta g(z) - \int_R \Delta_\Sigma G dxdy\right| &= \left|\sum_{\substack{vx_1wx_2\in F(\Lambda):\\ v,w\in V(\Gamma), v\in R, w\notin R}} 4\mathrm{area}(z'x_1x_2) + 4\mathrm{area}(R_\Lambda) - 4\mathrm{area}(R)\right| \\ &\leq 4 \cdot \left(\left(\mathrm{Const} \cdot \frac{\mathcal{E}r}{h}\right) \cdot (\mathrm{Const} \cdot \mathcal{E}h^2) + 4hr\right) \\ &= \mathrm{Const} \cdot \mathcal{E}^2 hr.\end{aligned}$$

This is the statement to show since $\max_{z\in R} |D^2G(z)| = 2$.

5. Let $G(z) = \mathrm{Re}(z^2)$. Let the notation again be given through Figure 2.4. For a quadrilateral $Q \in \mathrm{star}(v)$ donate by z'' the point such that

$$\mathrm{Re}\left((v-w)\left(z'' - \frac{v+w}{2}\right)\right) = 0 = \mathrm{Re}\left((x_2 - x_1)\left(z'' - \frac{x_2+x_1}{2}\right)\right)$$

holds. With a similar argument as before we get

$$\Delta g(v) = \sum_{Q\sim v} 2\int_{[Q,v]} \star dk,$$

where $k : V(\Lambda) \to \mathbb{R},\ k(z) = \mathrm{Re}(z(z''-v))$ depends on $Q \in \mathrm{star}(v)$ since z'' depends on Q. Similar to the previous case we calculate

$$\Delta g(v) = \sum_{Q\sim v} 2\int_{[Q,v]} \star dk = \sum_{Q\sim v} \mathrm{Im}((z''-v)(x_2-x_1)).$$

Hence, we get by canceling repeating terms

$$\begin{aligned}\sum_{v\in R\cap V(\Gamma)} 2\Delta g(v) &= \sum_{\substack{vx_1wx_2\in F(\Lambda):\\ v,w\in V(\Gamma),v\in R,w\notin R}} 2\mathrm{Im}(z''(x_2-x_1))\\ &= \sum_{\substack{vx_1wx_2\in F(\Lambda):\\ v,w\in V(\Gamma),v\in R,w\notin R}} \mathrm{Im}((2z''-(x_2+x_1))(x_2-x_1)),\end{aligned}$$

where the last equality holds because of

$$\sum_{\substack{vx_1wx_2\in F(\Lambda):\\ v,w\in V(\Gamma),v\in R,w\notin R}} (x_2^2-x_1^2)=0.$$

Furthermore, we have $\Delta_\Sigma G=0$. As before, we get $|z''-\frac{x_1+x_2}{2}|\leq \mathrm{Const}\cdot\mathcal{E}h$. Again, Lemma 2.6 concludes

$$\begin{aligned}\left|\sum_{z\in R\cap V(\Gamma)} 2\Delta g(z) - \int_R \Delta_\Sigma G dxdy\right| &\leq \sum_{\substack{vx_1wx_2\in F(\Lambda):\\ v,w\in V(\Gamma),v\in R,w\notin R}} |2z''-(x_2+x_1)|\cdot|x_2-x_1|\\ &\leq \mathrm{Const}\cdot\mathcal{E}^2hr.\end{aligned}$$

6. Let $G(z)=\mathrm{Im}(z^2)$. It holds $\mathrm{Im}(\alpha\beta)=\mathrm{Im}(\alpha)\mathrm{Re}(\beta)+\mathrm{Re}(\alpha)\mathrm{Im}(\beta)$ for each $\alpha,\beta\in\mathbb{C}$. Therefore, we find in this case a z''' such that

$$\Delta g(v)=\sum_{Q\sim v}\int_{[Q,v]}\mathrm{Re}((z'''-v)(x_2-x_1))$$

holds for each vertex $v\in V(\Lambda)\cap R$. The estimate (2.7) follows analogously to the previous case.

7. Let $D^kG(0)=0$ for $k=0,1,2$. For the classical Laplacian operator we estimate

$$\left|\int_R \Delta_\Sigma G dxdy\right|\leq r^2\cdot\max_{z\in R}|\Delta_\Sigma G(z)|\leq \mathrm{Const}\cdot r^3\max_{z\in R}|D^3G(z)|. \tag{2.10}$$

For the discrete Laplacian operator we get

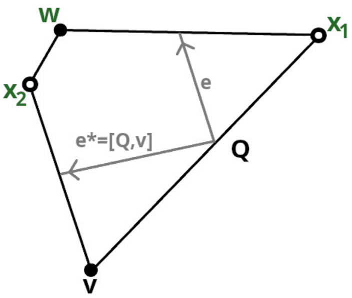

Figure 2.7: Use of Proposition 1.4

$$\left|\sum_{v\in R\cap V(\Gamma)} \Delta g(v)\right| = \left|\sum_{v\in R\cap V(\Gamma)} \oint_{P_v} \star dg\right| = \left|\sum_{\substack{Q=vx_1wx_2\in F(\Lambda):\\ v,w\in V(\Gamma),v\in R,w\notin R}} \int_{[Q,v]} \star dg\right|$$
$$\leq \mathrm{Const}\cdot \frac{\mathcal{E}r}{h}\cdot \max_{v\in R\cap V(\Gamma)} \left|\int_{[Q,v]} \star dg\right|,$$

where we again used Lemma 2.6. Let $v \in V(\Gamma) \cap R$. By using Proposition 1.4 with $e^* = [Q, v]$ and $e \in \vec{E}(X)$ given in Figure 2.7 and the definition of the eccentricity $\mathcal{E}$ in (2.1) we get

$$\begin{aligned}\left|\int_{[Q,v]} \star dg\right| &= \frac{1}{\sin\varphi_Q}\left|\frac{|x_2-x_1|}{|w-v|}\int_e dg - \cos\varphi_Q \int_{e^*} dg\right| \\ &\leq \mathcal{E}\left(\frac{|x_2-x_1|}{|w-v|}\left|\int_e dg\right| + \left|\int_{e^*} dg\right|\right) \\ &= \mathcal{E}\left(\frac{|x_2-x_1|}{|w-v|}\left|\frac{\nabla_Q g\cdot \vec{vw}}{2}\right| + \left|\frac{\nabla_Q g\cdot \vec{x_1x_2}}{2}\right|\right) \\ &\leq \mathcal{E}\left(\frac{2h}{|w-v|}\frac{|\nabla_Q g||w-v|}{2} + \frac{|\nabla_Q g|\cdot 2h}{2}\right) \\ &= \mathrm{Const}_{\mathcal{E}}\cdot h|\nabla_Q g|.\end{aligned}$$

For each $z \in R$ we have

$$|\nabla G(z)| \leq \mathrm{Const}\cdot r^2 \max_{z\in R}|D^3G(z)|, \quad |D^2G(z)| \leq \mathrm{Const}\cdot r \max_{z\in R}|D^3G(z)|.$$

Thus, and by use of Lemma 2.7 (Gradient Approximation Lemma), we get

$$\begin{aligned}|\nabla_Q g| &\leq |\nabla G(z) - \nabla_Q g| + |\nabla G(z)| \\ &\leq \mathrm{Const}\cdot\left(\mathcal{E}h\max_{z\in R}|D^2G(z)| + r^2\max_{z\in R}|D^3G(z)|\right) \\ &\leq \mathrm{Const}\cdot\left(\mathcal{E}hr + r^2\right)\max_{z\in R}|D^3G(z)| \leq \mathrm{Const}_{\mathcal{E}}\cdot r^2\max_{z\in R}|D^3G(z)|,\end{aligned}$$

where $Q \in F(\Lambda)$ is a face covering $z \in R$. Summarized we conclude

$$\left|\sum_{v\in R\cap V(\Gamma)} \Delta g(v)\right| \leq \mathrm{Const}_{\mathcal{E}}\cdot r^3 \max_{z\in R}|D^3G(z)|. \tag{2.11}$$

Because of (2.10) and (2.11) the statement holds.

□

Definition. Let $\Omega \subset \mathbb{C}$ be a connected domain. A sequence of quad-surfaced polygons $(\Sigma_n, \Lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ *approximates* Ω if the following properties hold as $n \to \infty$.

- The maximal distance from a point of $\partial\Lambda_n$ to the set $\partial\Omega$ tends to zero.
- The maximal distance from a point of $\partial\Omega$ to the set $\partial\Lambda_n$ tends to zero.
- The maximal edge length of the quad-surface Λ_n tends to zero.

Remark. Let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a nondegenerate uniform sequence of discrete Riemann surfaces such that the maximal edge length converges to zero as $n \to \infty$, and let $\Omega \subset \mathbb{C}$ be a bounded domain such that, by isometry, Ω is a subset of Σ. Therefore, we find for any $n \in \mathbb{N}$ (possibly empty) collections $Q_n \subseteq F(\Lambda_n)$ of faces within Σ. W.l.o.g let $\Sigma^{(n)} := \bigcup Q_n$ be either empty or a polygon quad-surfaced by Q_n. Hence, by the definition of nondegenerate uniform sequences of discrete Riemann surfaces, we get a sequence of quad-surfaced polygons $(\Sigma^{(n)}, \Lambda_n^{(n)})_{n\in\mathbb{N}}$ in $\mathbb{C}$ approximating Ω.

This observation will be helpful in the proof of Theorem 2.1 in Section 2.3, as well as the following lemma. To prove this, let us briefly recall the *Weyl lemma.*

Proposition 2.9 (Weyl [Wey40])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, let $\Delta_{\mathbb{R}^n}$ be the usual Laplacian operator on $\mathbb{R}^n$ and let $u \in L^1_{\text{loc}}(\Omega)$ locally integrable. If*

$$\int_\Omega u\Delta_{\mathbb{R}^n} f dx = 0$$

holds for any smooth function $f \in C_c^\infty(\Omega)$ with compact support, then u is harmonic.

Lemma 2.10. *Let $(\Sigma_n, \Lambda_n)_{n\in\mathbb{N}}$ be a nondegenerate uniform sequence of orthodiagonal quad-surfaced polygons approximating a domain $\Omega \subset \mathbb{C}$. Let $v_n : V(\Lambda_n) \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of discrete harmonic functions such that the sequence of restrictions to $V(\Gamma_n), n \in \mathbb{N}$, is uniformly converging to a continuous function $u : \Omega \to \mathbb{R}$. Then, u is harmonic.*
The same statement holds for Γ^* instead of Γ.

Before we prove the statement, let us make a short observation. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be both sequences in $[0, \infty[$ with $\lim_{n\to\infty} y_n = 0$ and

$$\limsup_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} \sup_{k\geq n} (x_k - y_k) =: y \leq 0. \tag{2.12}$$

Then, we get

$$0 \leq x_n \leq \sup_{k\geq n}(x_k - y_k) + y_n \xrightarrow[n\to\infty]{} y \leq 0, \tag{2.13}$$

and therefore the convergence of $(x_n)_{n\in\mathbb{N}}$ to zero as well. We denote property (2.12) by $x_n \preceq y_n$.

As previous, we are guided by [Sko13], Lemma 4.13.

Proof. Let $K \subset \Omega$ be an arbitrary compact set with smooth boundary and let $f : \Omega \to \mathbb{R}$ be any arbitrary smooth function vanishing outside of K. By Proposition 2.9 we have to prove that $\int_\Omega u\Delta_{\mathbb{C}} f dxdy = 0$ holds. To get this we show two things.

- The term

  $$\left| \int_\Omega u\Delta_{\mathbb{C}} f dxdy - \sum_{z\in V(\Gamma_n)} 2(v_n \Delta_n f)(z) \right| \tag{2.14}$$

  converges to zero as $n \to \infty$, where Δ_n describes the discrete Laplacian Operator on Λ_n.
- The discrete counterpart $\sum_{z\in V(\Gamma_n)} 2(v_n\Delta_n f)(z)$ in the difference (2.14) is zero for each $n \in \mathbb{N}$ large enough such that K is inside $\partial\Lambda_n$.[2]

---

[2] This statement is called the *discrete Weyl lemma.*

Let us start with the proof of the first statement. For each $n \in \mathbb{N}$ consider an auxiliary infinite square grid with edge length $r := \sqrt{2h}$, where $h := h_n$ is the maximal edge length of Λ_n. W.l.o.g. let $h_n < 1$ for each $n \in \mathbb{N}$. This ensures the assumption $r > 2h$ of the Laplacian Approximation Lemma. Further, we define for all $n \in \mathbb{N}$ functions $\hat{v}_n$ on the faces R of the n-th square grid, i.e., on the vertices of its dual grid, non-vanishing for $R \cap K \neq \emptyset$ through

$$\hat{v}_n(R) := \max_{z \in R \cap K} u(z).$$

Because of the continuity of u we get the uniformly convergence of $(\hat{v}_n)_{n \in \mathbb{N}}$ to u on the compact set K. By using both uniformly convergences we get

$$\left| \int_\Omega u \Delta_{\mathbb{C}} f dxdy - \sum_{z \in V(\Gamma_n)} 2(v_n \Delta_n f)(z) \right| \preceq \sum_{R: R \cap K \neq \emptyset} |\hat{v}_n(R)| \left| \int_R \Delta_{\mathbb{C}} f dxdy - \sum_{z \in R \cap V(\Gamma_n)} 2\Delta_n f(z) \right|. \tag{2.15}$$

More precisely, this is a consequence of

$$\begin{aligned}
&\left| \int_\Omega u \Delta_{\mathbb{C}} f dxdy - \sum_{z \in V(\Gamma_n)} 2(v_n \Delta_n f)(z) \right| \\
&= \left| \sum_{R: R \cap \Omega \neq \emptyset} \left(\int_R (\hat{v}_n + (u - \hat{v}_n)) \Delta_{\mathbb{C}} f dxdy - \sum_{z \in V(\Gamma_n) \cap R} 2((\hat{v}_n + (v_n - \hat{v}_n))) \Delta_n f)(z) \right) \right| \\
&\leq \sum_{R: R \cap \Omega \neq \emptyset} \left(|\hat{v}_n(R)| \left| \int_R \Delta_{\mathbb{C}} f dxdy - \sum_{z \in R \cap V(\Gamma_n)} 2\Delta_n f(z) \right| + \left| \int_R (u - \hat{v}_n) \Delta_{\mathbb{C}} f dxdy \right| + \left| 2 \sum_{z \in R \cap V(\Gamma_n)} ((v_n - \hat{v}_n)) \Delta_n f)(z) \right| \right).
\end{aligned}$$

The first summand is the right-hand side of (2.15). The other two terms converge to zero. On the one hand, the uniformly convergences on K deduce

$$\lim_{n \to \infty} \max_{z \in K} |u(z) - \hat{v}_n(R)| = 0, \quad \lim_{n \to \infty} \max_{z \in K \cap V(\Gamma_n)} |v_n(z) - \hat{v}_n(R)| = 0,$$

where R depends on n and covers z. On the other hand, we have seen in (2.10) and (2.11) in the proof of Lemma 2.8 that $|\int_R \Delta_{\mathbb{C}} f dxdy|$ and $|\sum_{z \in R \cap V(\Gamma_n)} (\Delta_n f)(z)|$ are bounded.

Now we use Lemma 2.8. For each $n \in \mathbb{N}$ we get

$$\begin{aligned}
\left| \int_R \Delta_{\mathbb{C}} f dxdy - \sum_{z \in R \cap V(\Gamma_n)} 2\Delta_n f(z) \right| &\leq \mathrm{Const}_{\mathcal{E}} \cdot (2h\sqrt{2h} \max_{z \in R} |D^2 f(z)| + 2h\sqrt{2h} \max_{z \in R} |D^3 f(z)|) \\
&\leq \mathrm{Const}_{\mathcal{E}} \cdot 4h\sqrt{2h} \max_{z \in K} \{|D^2 f(z)|, |D^3 f(z)|\}.
\end{aligned}$$

Moreover, we get

$$\sum_{R: R \cap K \neq \emptyset} |\hat{v}_n(R)| \leq \sum_{R: R \cap K \neq \emptyset} \left| \max_{z \in K} u(z) \right| \leq \mathrm{Const} \cdot \frac{\mathrm{area}(K)}{2h} \cdot \left| \max_{z \in K} u(z) \right|,$$

because $2h$ is the area of R. Therefore, in (2.15) we get

$$\begin{aligned}
\left| \int_\Omega u \Delta_{\mathbb{C}} f dxdy - \sum_{z \in V(\Gamma_n)} 2(v_n \Delta_n f)(z) \right| &\preceq \mathrm{Const}_{\mathcal{E}} \frac{\mathrm{area}(K)}{2h} \cdot \left| \max_{z \in K} u(z) \right| \cdot 4h\sqrt{2h} \max_{z \in K} \{|D^2 f(z)|, |D^3 f(z)|\} \\
&\leq \mathrm{Const}_{\mathcal{E},K,u,f} \cdot 4h \xrightarrow[n \to \infty]{} 0.
\end{aligned}$$

Hence, we get the first statement by using (2.13).

Now we show the second statement. Let $n_0 \in \mathbb{N}$ be large enough such that K is inside each $\partial\Lambda_n$ for $n \geq n_0$. Therefore, we have $f(z) = 0$ for each $z \in \partial\Lambda_n$. By also using the harmonicity of v_n, we get $(f\Delta_n v_n)(z) = 0$ for each $z \in V(\Lambda_n)$. Hence, Green's Identity (1.5) concludes

$$\sum_{z\in V(\Gamma_n)} (v_n\Delta_n f)(z) = \sum_{z\in V(\Gamma_n)} (f\Delta_n v_n)(z) = 0,$$

because the right-hand side of Corollary 1.9 vanishes since f is compactly supported.
Therefore, we get $\int_\Omega u\Delta_{\mathbb{C}} f dxdy = 0$. □

2.3 Proof of the convergence theorem

Now, we are ready to prove the convergence of the sequence of discrete harmonic function given in Theorem 2.1. Let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a nondegenerate uniform sequence of orthodiagonal discrete Riemann surfaces such that the maximal edge length converges to zero as $n \to \infty$.

We need a variant of the *Arzelà-Ascoli* theorem.

Definition. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of real multi-valued functions on discrete cones $(\Sigma_k, \Lambda_k)_{k\in\mathbb{N}}$ of (Σ, Λ) with periods $(P_k)_{k\in\mathbb{N}}$. We call the sequence $(f_k)_{k\in\mathbb{N}}$ *uniformly bounded* if there is a constant C depending only on Σ, Λ and $(P_k)_{k\in\mathbb{N}}$ such that

$$|f_k(z)| < C$$

holds for any $k \in \mathbb{N}$ and $z \in V(\Lambda_k)$.
We call this sequence *equicontinuous* if there is a positive function $\delta(\varepsilon)$ not depending on k such that for any $z, w \in V(\Lambda_k)$ with $|zw| < \delta(\varepsilon)$ we have

$$|f_k(z) - f_k(w)| < \varepsilon.$$

Theorem 2.11 (Arzelà-Ascoli [Kel91])**.** Let $K \subset \widetilde{\Sigma}$ be a compact subset and let $f_k : V(\Gamma_k) \cap K \to \mathbb{R}$, $k \in \mathbb{N}$, be an equicontinuous and uniformly bounded sequence. Then, there exist a continuous function $f : K \to \mathbb{R}$ and a subsequence $(k_l)_{l\in\mathbb{N}} \subseteq \mathbb{N}$ such that $(f_{k_l})_{l\in\mathbb{N}}$ converges uniformly to f in K.

Proof of Theorem 2.1. Let $(\widetilde{\Lambda}_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(\widetilde{\Lambda}_n)_{n\in\mathbb{N}}$. For each $k \in \mathbb{N}$ we denote for simplicity $\widetilde{\Lambda}_k := \widetilde{\Lambda}_{n_k}$. Take a sequence of compact sets $K_1 \subset K_2 \subset ... \subset \widetilde{\Sigma}$ such that $\widetilde{\Sigma} = \bigcup_{j=1}^{\infty} K_j$. Further assume that K_1 contains the converging subsequence $(p_k, p'_k)_{k\in\mathbb{N}}$ of edges to $p \in \Sigma$. By assumption all face angles of each quadrangulation are bounded from below by a constant $\Phi > 0$. Let $10r'$ be either the minimal distance of the conical singularities of $\widetilde{\Sigma}$ or the diameter of Σ if there are no singularities. Let $v \in \widetilde{\Sigma}$ be a conical singularity or at distance of at least $5r'$ to a singularity. Define

$$\Sigma'_v := \{x \in \Sigma \mid |xv| \leq 3r'\} \subset \Sigma.$$

Since the maximal edge length converges to zero, take k_1 such that for each $k > k_1$ the maximal edge length of $\widetilde{\Lambda}_k$ is less than $\frac{r' \sin\Phi}{12\gamma'_v}$. Note that $\frac{\sin\Phi}{12\gamma'_v} \leq \frac{1}{12}$ holds since $\gamma'_v \geq 1$.
Fix $k > k_1$. Let $\Sigma_v \subset \Sigma$ be the union of faces in $F(\widetilde{\Lambda}_k)$ within distance at most $4r'$ from the vertex v. Therefore, we get a discrete cone (Σ_v, Λ_v) around v where Λ_v is the restriction of $\widetilde{\Lambda}_k$ to Σ_v.

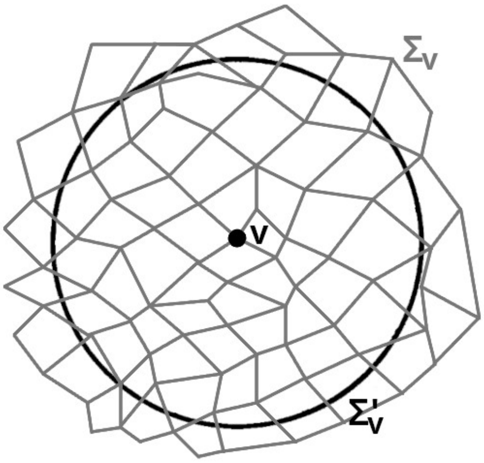

Figure 2.8: Disc Σ'_v of radius $3r'$ and discrete cone (Σ_v, Λ_v) with $\mathrm{dist}(v, \partial\Sigma_v) \leq 4r'$ around $v \in \widetilde{\Sigma}$.

Now, we estimate the right-hand side of the inequality in Lemma 2.5 (Equicontinuity Lemma) for the discrete harmonic function $\nu := v_k|_{V(\Lambda_v)}$ and same coloured $z, w \in \Sigma'_v \cap V(\Lambda_v)$ with $|zw| < \frac{r'}{12\gamma'_v}$. W.l.o.g. we consider $\nu = v_k|_{V(\Gamma_v)}$ and always black vertices, because the same arguments fit for Γ^*. Since

$$r = \mathrm{dist}(v, \partial\Sigma_v) - \max\{|vz|, |vw|\} - 2h > \mathrm{dist}(v, \partial\Sigma_v) - 3r' - \frac{\sin\Phi}{6\gamma'_v} r' > \frac{1}{2} r',$$

it follows

$$\frac{r}{3\gamma'_v |zw|} > \frac{r}{3\gamma'_v \frac{r'}{12\gamma'_v}} = \frac{r}{\frac{1}{4} r'} > \frac{2r}{r} = 2,$$

thus $r > 3\gamma'_v |zw|$. In other word, the Equicontinuity Lemma works with those chosen vertices z, w. By Theorem 1.12 the sequence $(E_{\Lambda_v}(v_k))_{k \in \mathbb{N}}$ is bounded. Thus, by Lemma 2.5, for each $k > k_1$, the function $v_k|_{\Sigma'_v \cap V(\widetilde{\Gamma}_k)}$ has *uniformly bounded differences*, i.e., there is a constant $C := \mathrm{Const}_{\Sigma, (\Lambda_n)_n, (P_n)_n}$ such that

$$|v_k(z) - v_k(w)| < C$$

holds for any $z, w \in \Sigma'_v \cap V(\widetilde{\Gamma}_k)$. By the same lemma the sequence $(v_k|_{\Sigma'_v \cap V(\widetilde{\Gamma}_k)})_{k > k_1}$ is equicontinuous.

Now cover the compact set K_1 by finitely many sets Σ'_v, where $v \in \widetilde{\Sigma}$ is either a conical singularity or at distance of at least $5r'$ to a singularity, such that sufficiently many pairs of these sets have an intersection containing an intrinsic disk of radius more than $\frac{r'}{4}$, i.e., these disks contain black vertices. Therefore, the uniformly bounded differences (depending also on the number of sets covering K_1) and the equicontinuity remains true.

Moreover, by using $(p_k)_{k > k_1} \subset K_1$ and $v_k(p_k) = 0$, the sequence is uniformly bounded, too. Hence, the Arzelà-Ascoli theorem gives us a continuous function $u_1 : K_1 \to \mathbb{R}$ and a subsequence $(l_\kappa)_{\kappa \in \mathbb{N}} \subseteq \mathbb{N}$ such that $l_1 = k_1$ and $(v_{l_\kappa}|_{V(\Gamma)})_{\kappa \in \mathbb{N}}$ converges to u_1 uniformly in K_1.

Analogously, we get a continuous function $u_2 : K_2 \to \mathbb{R}$ and a subsequence $(m_\kappa)_{\kappa \in \mathbb{N}} \subseteq (l_\kappa)_{\kappa \in \mathbb{N}}$ such that $m_1 = l_1 = k_1$, $m_2 = l_2$ and $(v_{m_\kappa}|_{V(\Gamma)})_{\kappa \in \mathbb{N}}$ converges to u_2 uniformly in K_2. It is $u_1 = u_2$ on K_1. We continue this process to get a continuous function $u_0 : \widetilde{\Sigma} \to \mathbb{R}$ and a subsequence $(q_\kappa)_{\kappa \in \mathbb{N}} \subseteq \mathbb{N}$ such that $(v_{q_\kappa}|_{V(\Gamma)})_{\kappa \in \mathbb{N}}$ converges to u_0 uniformly on each compact subset of $\widetilde{\Sigma}$.

Clearly, the periods of u_0 are given by the periods P of u. To see this, we consider any deck transformation d_γ of $\widetilde{\Sigma}$, an arbitrary sequence $(z_k)_{k \in \mathbb{N}} \subseteq V(\widetilde{\Lambda}_{q_k})$ converging to a $z_0 \in \widetilde{\Sigma}$ and the convergence of the equation

$$P_\gamma(v_{q_k}) = v_{q_k}(d_\gamma z_k) - v_{q_k}(z_k)$$

to

$$P_\gamma(u) = u_0(d_\gamma z_0) - u_0(z_0)$$

for $k \to \infty$, where $P_\gamma(v_{q_k}),\ P_\gamma(u) \in \mathbb{R}$ are the to $\gamma \in H_1(\Sigma, \mathbb{Z})$ corresponding periods given by $P_{q_k},\ P \in \mathbb{R}^{2g}$. By the same argument we get $u_0(p) = 0$.

Now, we want to use Lemma 2.10 to get the harmonicity of u_0. For each edge $e \in E(\widetilde{\Lambda}_1)$ let $\Omega_e \subset \widetilde{\Sigma}$ be the union of e without endpoints and the interior of the two faces of $F(\widetilde{\Lambda}_1)$ at e. We can assume $\Omega_e \subset \mathbb{C}$ by using an isometry. For each $k \in \mathbb{N}$ let $\Lambda_k^{(e)}$ be the sub-quad-graph of Λ_{q_k} including exactly the faces within Ω_e. Therefore, we get a sequence of quad-surfaced polygons approximating Ω_e, and by Lemma 2.10 the harmonicity of u_0 on

$$\bigcup_{e \in E(\widetilde{\Lambda}_1)} \Omega_e \subset \widetilde{\Sigma}.$$

This union excludes all points in $V(\widetilde{\Lambda}_1)$ (in particular all conical singularities of $\widetilde{\Sigma}$). It is u_0 a continuous function on $\widetilde{\Sigma}$, therefore, by the *singularity removal theorem*, we get the harmonicity on $\widetilde{\Sigma}$. Thus $u_0 = u$ normalized at p.

Because we can start the proof with any arbitrary subsequence of $(\widetilde{\Lambda}_n)_{n\in\mathbb{N}}$ to get the same unique limit function u_0, it follows that the whole sequence $(v_n|_{V(\Gamma)})_{n\in\mathbb{N}}$ converges to u uniformly on each compact subset. In the same way the sequence $(v_n|_{V(\Gamma^*)})_{n\in\mathbb{N}}$ converges to u uniformly on each compact subset, and therefore, also the sequence $(v_n)_{n\in\mathbb{N}}$ converges to u uniformly on each compact subset since

$$\max_{z \in V(\widetilde{\Lambda}_n)\cap\Omega} |v_n(z) - u(z)| \leq \max_{z \in V(\widetilde{\Gamma}_n)\cap\Omega} |v_n(z) - u(z)| + \max_{z \in V(\widetilde{\Gamma}_n^*)\cap\Omega} |v_n(z) - u(z)|$$

holds for all $n \in \mathbb{N}$. □

Chapter 3

Convergence of discrete holomorphic integrals

In this chapter we go a step forward and prove the convergence of holomorphic integrals over a nondegenerate uniform sequence of orthodiagonal discrete Riemann surfaces.

Definition. Let ω be a (discrete) holomorphic one-form. We call the primitive $\int \omega$ *(discrete) holomorphic integral* or *(discrete) Abelian integral of the first kind*[3].

Theorem 3.1. Let $(\Sigma, \Lambda_n)_{n\in\mathbb{N}}$ be a nondegenerate uniform sequence of orthodiagonal discrete Riemann surfaces such that the maximal edge length converges to zero as $n \to \infty$. Let $\mathcal{A} \in \mathbb{C}^g$ be a given vector of a-periods and let $(p_n, p'_n)_{n\in\mathbb{N}} \subseteq E(\Lambda_n)$ be a sequence of edges such that $p_n \to p \in \Sigma$ as $n \to \infty$. Denote by ω the unique holomorphic one-form with a-periods given by $\mathcal{A}$ and denote by $\omega_n,\ n \in \mathbb{N}$, the unique discrete holomorphic one-form on the medial graph of Λ_n with equal black and white a-periods given by $\mathcal{A}$. We normalize the holomorphic integral $\int \omega$ to attain the value zero at p, and $\int \omega_n$ shall be normalized in such a way that $\int \omega_n(p_n) = \int \omega_n(p'_n) = 0$.
Then, the discrete holomorphic integral $\int \omega_n$ converge to $\int w$ uniformly on each compact subset of $\widetilde{\Sigma}$.

As in [BS16], the statement is a direct consequence of Theorem 2.1 if we have the convergence of the black and white b-periods of the discrete harmonic functions to the b-periods of the smooth harmonic function. For orthodiagonal discrete Riemann surfaces, Corollary 1.14 delivers this convergence.

3.1 Proof of the convergence theorem as corollary of Theorem 2.1

Let the notation be given by Theorem 3.1. For each $n \in \mathbb{N}$ let the black and white b-periods of ω_n be denoted by $\mathcal{B}_n^B, \mathcal{B}_n^W \in \mathbb{C}^g$. Denote the b-periods of ω by $\mathcal{B} \in \mathbb{C}^g$. To use Theorem 2.1, we need the following convergence of the b-periods.

Lemma 3.2. It holds $\lim_{n\to\infty} \mathcal{B}_n^B = \mathcal{B} = \lim_{n\to\infty} \mathcal{B}_n^W$.

Proof. The statement is a direct consequence of Corollary 1.14, since, in the orthodiagonal case, $\mathrm{Re}(\mathcal{B}_n^B)$ and $\mathrm{Re}(\mathcal{B}_n^W)$ converge to $\mathrm{Re}(\mathcal{B})$, and $\mathrm{Im}(\mathcal{B}_n^B)$ and $\mathrm{Im}(\mathcal{B}_n^W)$ converge to $\mathrm{Im}(\mathcal{B})$. ☐

We denote the (discrete) holomorphic multi-valued functions by

$$\int \omega =: f = \mathrm{Re}(f) + i\mathrm{Im}(f), \quad \int \omega_n =: f_n = \mathrm{Re}(f_n) + i\mathrm{Im}(f_n),\ n \in \mathbb{N},$$

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[3] More about discrete Abelian integrals of the first, second and third kind can be read in [BG17], Chapter 7.

where the real and imaginary parts are (discrete) harmonic multi-valued functions with real a-periods given by $\mathrm{Re}(\mathcal{A})$ and $\mathrm{Im}(\mathcal{A})$, respectively.
By Lemma 1.10 there exist (discrete) harmonic one-forms $\alpha, \beta, \alpha_n, \beta_n,\ n \in \mathbb{N}$, such that

$$\mathrm{Re}(f) = \int \alpha,\ \mathrm{Re}(f_n) = \int \alpha_n, \quad \mathrm{Im}(f) = \int \beta,\ \mathrm{Im}(f_n) = \int \beta_n,$$

normalized at p or p_n, p'_n, respectively, with a-periods given by $\mathrm{Re}(\mathcal{A})$ and $\mathrm{Im}(\mathcal{A})$.

Therefore, we only need to prove the statement for given real vectors $\mathcal{A}, \mathcal{B}, \mathcal{B}_n^B, \mathcal{B}_n^W \in \mathbb{R}^g, n \in \mathbb{N}$, a (unique) harmonic one-form ω with a-periods given by $\mathcal{A}$ and b-periods given by $\mathcal{B}$, and (unique) discrete harmonic one-forms $\omega_n,\ n \in \mathbb{N}$, with equal black and white a-periods given by $\mathcal{A}$ and black and white b-periods given by $\mathcal{B}_n^B, \mathcal{B}_n^W$.

Proof of Theorem 3.1. Let $u : \widetilde{\Sigma} \to \mathbb{R},\ v_n : V(\widetilde{\Lambda}_n) \to \mathbb{R},\ n \in \mathbb{N}$, be the (discrete) harmonic multi-valued functions $u = \int \omega$ and $v_n = \int \omega_n$ normalized at p or p_n, p'_n, respectively, with periods given by $\mathcal{A}, \mathcal{B}, \mathcal{B}_n^B, \mathcal{B}_n^W \in \mathbb{R}^g$. By Lemma 3.2 we get

$$\lim_{n\to\infty} \mathcal{B}_n^B = \mathcal{B} = \lim_{n\to\infty} \mathcal{B}_n^W .$$

Thus, by Theorem 2.1 $(v_n)_{n\in\mathbb{N}}$ converges to u uniformly on each compact subset of $\widetilde{\Sigma}$. □

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